Notes on Measurement Errors

(Some material in this handout excerpted from Stan Micklavzina's "Guidelines for Reporting Data" and from William Lichten's "Data and Error Analysis," Allyn and Bacon)

Measurements are Central to Science. The laws of science are discovered through measurements. They are hypothesized from a related set of real-world measurements. They are verified and refined by means of critically designed measurements. Any law that has been contradicted by even a single measurement must be discarded immediately and be replaced by another. Measurements are ***the final authority*** in science. This paradigm has been an essential ingredient for the development of science in Europe starting about 500 years ago. The centrality of measurements remains unaltered in science today. As scientists, we **must** learn all we can about measurements.

Measurements are Approximate. Let's suppose you measured the length of your pencil with a ruler. It is incorrect for you to claim, "My new pencil is exactly 192 millimeters long." If you were to use a more exact measuring device you might say, "Oops! My pencil is 192.16 millimeters long." Your first measurement is good to the nearest millimeter; your second is good to the nearest 0.01 mm. We say that both values are inexact or approximate; both are subject to measurement *uncertainties* (or errors). The rest of this note discusses these uncertainties and how they affect our confidence in our own measurment results.

Mistakes Versus Errors. The word "error" has a special non-colloquial meaning in science. Error is different from mistake. Mistakes, such as measuring a 32-cm-long object to be 42 cm, can be avoided. As we shall see, errors cannot be avoided, even by the most careful measurements. Hence, errors quantify the degree of confidence we have in the associated measurements.

Precision Versus Accuracy: Random and Systematic Errors. Let's go back to the example of the pencil. Suppose everyone in the class uses the same ruler, measures the pencil to the nearest millimeter, and all agree it is 192 mm long. All say that it couldn't be either 191 or 193 mm long. We say that the class has measured the length of the pencil to a ***precision*** of 1 mm. Precision is the reliability or repeatability of a measurement. Suppose that the instructor now points out, "You all have made the same mistake. You lined up one end of the pencil and one end of the ruler together. The end of the ruler is worn badly; it doesn't begin at zero. Try to remeasure the pencil by putting it in the middle of the ruler. Then find the position of both ends." (see Table 1 below.) "Subtract one value from the other to find the length." Now the class finds that the pencil is 187 mm long! How can this be? Both measurements are equally precise. The second one is more ***accurate*** than the first, because a systematic error (caused by the worn end of the ruler) is no longer there. A ***systematic error*** is an effect that changes all measurements by the same amount or by the same percentage. The class's experience with the ruler is a mirror of the history of science. Systematic errors have often crept unsuspectedly into measurements. The only way to eliminate systematic errors is to look carefully for them and to understand well the nature of the experiment or measurement.

TABLE 1 Measurement of the Length of a Pencil.

Left End (cm)	Right End (cm)	Length (cm)	Deviation from Mean
10.16	28.83	18.67	-0.03
15.87	34.57	18.70	0.00
20.22	38.95	18.73	+0.03

<u>Random Errors: We Can Not Avoid Them.</u> Let's return to the example of the class measurement of the length of a pencil; when measuring to the nearest millimeter, everyone got the same value. Let's try to push the precision further and ask each person to measure to the nearest tenth of a millimeter. Now disagreements appear. We find different values: 186.7, 187.0, 187.3 mm, as shown. Is someone making a mistake? No, even the most careful and skillful person will come up with values that vary by one- or two-tenths of a millimeter. Now we are at the limit of measurement by use of the naked eye and rulers. The unavoidable change in successive measurements, due to small irregularities in the ruler, difficulty in estimating precisely, and the like, is called a ***random error***, or ***error*** for short.

<u>Your Best Estimate.</u> Thus far, you have been careful not to make any mistakes, you have avoided all systematic errors, and you have narrowed your uncertainty to the random error of measurement. What's next? Common sense tells you to take the average of several measurements, called the ***arithmetic mean*** or ***mean.*** The algebraic expression for the

average of N numbers is

$$\overline{x}=\frac{1}{N}\sum_{i=1}^{N}x_i=\frac{x_1+x_2+\cdots+x_N}{N}$$

The data scatter gives you an idea of the random error of measurement. A handy measure is the ***average deviation from the mean***, sometimes shortened to ***average deviation.*** You can get this by finding the difference between each measurement and the mean and then taking the average. (You count all deviations as positive for this calculation.)

$$Average\ deviation=\frac{1}{N}\sum_{i=1}^{N}|x_i-\overline{x}|$$

The final result is 18.70(2) cm = 18.70 ± 0.02 cm. Note two ways of showing the error: the symbol ± precedes the error, or parentheses show the error in the last place. You will learn later that, if you take three measurements, the average deviation is a remarkably good estimate of the error of your measurement. Let's go over this again: ***Take three measurements. Take the average as your best estimate of the true value. Take the average deviation as an estimate of the error of measurement.*** This is a good rule of thumb that has several advantages. It's simple. It's easy to do the calculations; most of the time you can do them in your head or on a very small piece of paper.

<u>Relative and Percentage Errors.</u> So far, an error has had the same units as the measured quantity: it has been so many millimeters or so many grams. (Sometimes the name *absolute error* is used in this connection.) If an astronomer gave the distance to the moon to the nearest meter, we would consider it a breathtaking triumph of a measurement project. But if you wanted to order a ball bearing, you would need to know the shaft diameter to a very small fraction of a millimeter. Absolute errors should be compared with the measured quantity always. In scientific measurements, it often is meaningful to express errors in fractions or per cents.

$$Relative\ error\ in\ a\ quantity=\frac{error}{measured\ quanity}$$

$$Percentage\ error\ in\ a\ quantity=\frac{100\times error}{measured\ quantity}$$

Example: Errors in Measuring with a Meter Stick
Discuss the two sources of error of measurement with a meter stick. The first occurs because 1 mm is worn off the zero end of the stick. The second is due to a uniform shrinking of the meter stick over its entire length by 1 mm. In particular, calculate the different types of errors for measuring two objects: one

is reported to be 0.999 m long; the second is reported to be 10 cm long. Solution. The worn end of the ruler causes the same error. No matter what the length of the object, it will appear to be 1 mm longer than its true value. On the other hand, the uniform shrinkage of the meter stick causes the same fractional or percentage error. We first note that the meter stick is actually 999 mm long. The 999-mm-long object would appear to be 1 m long and the error in measuring the length would be 1 mm. The fractional error would be (1 mm / 999mm). The percentage error would be 0.1%. For the short object, the worn end causes an error of 1 mm, a fractional error of (1 mm / 100 mm = 0.01), and a percentage error of 1%. The shrinkage of a 10 cm length of the ruler is only one-tenth of the shrinkage of 1 m. Thus the error is 0.1 mm. The fractional error is (0.1 mm / 10 cm) = 0.001, the same as for the long object. The percentage error is again 0.1%. The uniform shrinkage or expansion of an meter stick or any other scale causes the same fractional or percentage error.

Significant Figures.

> ***RULE: Experimental values — both measurements and results calculated from these measurements — should always be reported with only one uncertain digit.***

As you have seen in previous discussions, the right-most digit of any experimental measurement contains an error. "Significant figures" are those digits which have significance — which have meaning. If you had measured the length of this page with a mm—ruler and given the result as 279.33 mm, you would be incorrect. The smallest division is a mm, and you can estimate one more digit corresponding to a place between two divisions. You have absolutely no information on hundredths of a mm. That is, in this measurement, the tenths place is somewhat uncertain but the hundredths place is completely uncertain — it has no experimental significance. So your reported value should be 279.3 but not 279.33, or 279.

Things were simple before calculators. If you carried out your calculations with a slide rule (an antiquated hand calculator known only to persons born before 1955) you would be limited to three figures. If you did them by hand, you would be only too happy to round off. But, because at the touch of a button you have eight or nine digits displayed before your eyes, you will have difficulty with this simple rule. The calculator makes life difficult because you must decide which of these digits are uncertain, and you must round off all but one uncertain digit. The rules for rounding off are as follows:

- Examine the digits to be discarded.
- If the first digit is larger than a 5, round up.
- If the first digit is less than a 5, round down.
- If the first digit is a 5 followed by other digits, at least one of which is not zero, round up.
- If the first digit is 5 followed only by zeros, or by no other digits at all, round up or down to make the last digit retained even. [For example, 2.55 is rounded to 2.6, while 2.45 is rounded to 2.4.]

This business of "significant figures" is the simplest of the *error analysis*, which you will try to master throughout your scientific career. For the moment, let's look at a few examples to establish useful rules about significant figures. To keep track of uncertain digits, they will be emboldened and underlined in these examples. Suppose you have the object made up of three separate component parts — a ball, a cylinder and a plate. You measure the masses by using a precision analytical balance for the ball and the cylinder and a triple beam balance for the plate. The masses are .28***2*** gm, 79.54***5*** gm and 422.2***3*** gm. The total mass is

$$\begin{array}{r} 0.28\underline{\mathbf{2}} \\ 79.54\underline{\mathbf{5}} \\ 422.2\underline{\mathbf{3}} \\ \hline 502.0\underline{\mathbf{57}} \end{array}$$

The result as written is incorrect — it has two uncertain digits. To conform to the rule, the mass must be rounded up, and be written as 502.06 gm. Note that this isn't at all profound! What we are saying in this example is that if you can only measure the mass of one part of the object to a precision of hundredths of a gram, you cannot possibly know the mass of the composite object to a precision of thousandths of a gram. The general rule for addition, which works as well for subtraction is: ***When calculating an experimental result by adding or subtracting experimental data, round off so only one uncertain digit remains in the result.***

Now suppose we want to find the area of a rectangle. The length, measured with a meter stick, is found to be 11.23 cm and the width, measured with a vernier caliper, is found to be .332 cm. The area is

$$\begin{array}{r} 11.2\underline{3} \\ \text{X}\quad .33\underline{2} \\ \hline \underline{2246} \\ 336\underline{9} \\ 336\underline{9} \\ \hline 3.7\underline{2836} \end{array}$$

The correct result is 3.73 cm^2. Notice that the result has 3 significant digits--the same number as in the width. This result illustrates the following rule of thumb: ***When calculating an experimental result by multiplying or dividing experimental data, round off the result so that it has only as many significant figures as the data value with the smallest number of significant figures.***

Another quasi-rule often used by practitioners is: ***Add a significant digit when the first digit is a 1.*** Thus 3 x 0.34 = 1.02, not 1.0.

One last but important point. Consider a length 35.9 m, which can also be written as 3590 cm. But these two numbers have very different meanings. The first says that the 9 is uncertain while the second implies that only the last 0 is uncertain. If it is the 9 which is uncertain, how should we write the result properly in cm? The answer is $3.59 \text{ X } 10^3$ cm. Note that there are now only three significant figures.

<u>Propagation of Errors: Small refinement of the SigFig.</u> When measured quantities are combined (i.e., added or multiplied together), the rules associated with significant figures enabled us to make sensible statements about the resulting quantity. Consider the area of the rectangle treated above, which was 3.73 cm^2. The most conservative position we can take about this number is that the area has a value somewhere within the range bound by 3.700 and 3.800. Frequently, <u>the uncertainty is smaller than that indicated by the rules of</u>

significant figures. The question is, “Is there a simple way to track the error propagation?” The answer is yes. Again, the simple rules we will develop here mirror more rigorous results which you will learn later.

Let us calculate the area of a square, whose side is measured to be L = 6.71 ± 0.02 cm. Let's write this as

$$L = 6.71 \times [1 \pm \delta],$$

where the bracketed quantity is a mathematical construct, not a measured quantity. Here, δ must be equal to (0.02 / 6.71). We get, upon squaring,

$$A = L^2 = (6.71)^2 \times [1 \pm \delta] \times [1 \pm \delta],$$

where we note that the first and the second δ are completely correlated. In fact they are one and same. Hence,

$$A = 45.0 \times [1 \pm 2\delta + O(\delta^2)].$$

This should remind you a lot of the beginning of the differential calculus. In fact the *error analysis* does deal with negligibly small quantities in the same way the calculus does. In the following discussion, our measurement errors will be referred to as standard errors and are denoted by the variable σ with appropriate suffixes. You will learn the exact relationship between our measurement errors and standard errors later.

Propagation of Errors: Single Measurement. Let us assume that we made a measurement of an x and ask how the standard error σ_x propagates as different functions of x are computed. It is convenient to consider not only the error itself, but also the relative (fractional) error (σ_x / x). In error theory, we always consider the fractional error to be small compared to 1; i.e., (σ_x / x) << 1. (Large fractional errors are very unusual in physics laboratories.) All expressions that follow are based on this assumption.

Consider an arbitrary function ***z(x)***. We wish to know σ_z. Applying the concept learned in the analytical geometry, we obtain,

$$\frac{\sigma_z}{\sigma_x} = \frac{dz}{dx} \quad or \quad \sigma_z = \frac{dz}{dx}\sigma_x.$$

Now, suppose that $z(x)$ of the form,

$$z = ax^n \quad then \quad \sigma_z = nax^{n-1}\sigma_x$$

Manipulating further,

$$\sigma_z = nax^{n-1}\sigma_x\left(\frac{x}{x}\right) = zn\left(\frac{\sigma_x}{x}\right) \quad or \quad \frac{\sigma_z}{z} = n\frac{\sigma_x}{x}$$

This specially simple relationship between two relative errors exists only for this particular functional form. It is useful nevertheless, because so many error tracking operations in practice involve this type of functions (e.g. unit conversions). For other functional forms, one must go back to the general expression listed above.

Suppose you have a tiny postal scale that only weighs to 1 ounce, and you want to mail a ream (500 sheets) of paper. How much does it weigh? You cleverly take a packet of five sheets of paper, put them together, and find the total weight to be 1 ounce. Under the assumption that all sheets of paper have the same weight, the weight of a ream is 100 oz, or 6 lb and 4 oz. You put on enough postage for a 7-lb package. Will your package make it? That depends on the error of your estimate of the weight. Suppose your weighing had an error of 0.1 oz. Then the ream would have an error of 100 x 0.1 oz = 10 oz. At most, your package would weight 6 lb 14 oz, and you are safe. In this example you multiplied your measured quantity and its error by 100. You can readily imagine the reverse, in which you have only a large set of scales, weigh a ream of paper in pounds, and find the weight of a five-sheet letter by division. In this case, you would divide both your result and error by 100. To summarize: **When a measured quantity is multiplied (divided) by a constant, the absolute error is likewise multiplied (divided) by the same constant.**

Propagation of Errors. More Than One Measurement. When we started the discussion of error propagation, we considered an example of calculating the area of a square. In that example, we took one single measurement, and squared the data. Because δ's were completely correlated, the error doubled. What happens if we add, subtract, multiply or divide two independently measured quantities? The answer is not simple. But it is clear that those δ's are not correlated. In fact there is a probability that they may partially cancel each other out. You will learn in Physics 353 the problem of "drunkard's walk" which involves the treatment of uncorrelated errors. We will borrow the result from that treatment.

When two or more independent measurements are combined, there are four simple rules to remember. They are;

> **Rule #1: When two measurements are combined by addition or subtraction, use absolute errors, and use the recipe** - Given $z = x \pm y$ with x, σ_x, y, σ_y, then
> $\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$ (A cocktail party phrase.. *In addition of measured quantities, absolute errors are added in quadrature.*)

Rule #2: When two measurements are combined by multiplication or division, use relative errors, and use the recipe - Given $z = axy$ *or* $z = ax/y$, then

$$\frac{\sigma_z}{z} = \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2}$$ *(..In a product of measured quantities, relative errors add in quadrature.)*

Rule #3: Keep two most significant digits as errors are propagated. For the final answer, keep only the most significant digit in the absolute error.

Rule #4: The-factor-of-two-rule. *Remember that errors usually aren't more precise than ± 50%; one significant figure is all that you can expect in your error estimate. Given this, we will now see that, if one source of error, A, is appreciably larger than another source of error, B, then B has a negligible effect on the final error. Suppose A is equal to a 2% error and B is half as large. Then the final error estimate is*

$$\%\ error = \sqrt{(1^2 + 2^2)} = 2.24\ \% \approx 2\ \%.$$

To one significant figure, the total error is described completely by the larger source A. The conclusion is quite general: a successful error analysis finds the largest source of error, whether it be systematic or random, rather than attempting to add the effects of many small errors.

Propagation of Errors: Final Generalization. If the four propagation rules, listed above, become obscure in a very complicated error tracking situation, you may be forced to evaluate the following general formula:

$$If\quad z = z(x, y) \quad then \quad \sigma_z = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2 \sigma_y^2}$$

When there are more than two measured quantities, you can extend this expressions by adding more terms under the square-root sign.

Example: Error Propagation
A particle slides down an inclined plane starting from rest. We will predict the distance traveled by the particle in two seconds, using measured quantities. From our physics knowledge, we know the distance to be

$$L = \frac{g}{2}(\sin\alpha - \mu\cos\alpha)t^2.$$

We measured μ = 0.20(1), α = 0.52(2) radians, t = 2.0(1) and g = 980.5(2) cm /s^2. Solution using four rules listed. First, the predicted distance. My calculator displays 634.0232681 cm. Let us keep one more digit than what the rules about the significant figures say; 6.34 x 10^2 cm. To understand the propagation of errors, let us first calculate % errors. They are: 5% for μ, 4% for α, 5% for t, and 0.02% for g. Clearly, we need not worry about the error in g. (Rule #4)

$$\text{Error in } \sin\alpha = \sigma_\alpha \cos\alpha = 0.017$$
$$\text{Error in } \cos\alpha = \sigma_\alpha \sin\alpha = 0.01 \rightarrow 1.1\%$$
$$\text{Error in } \mu\cos\alpha = 5\% \ (\text{from } \mu) \rightarrow 0.0087$$
$$\text{Error in paran} = 0.017 \rightarrow 5.3\%$$
$$\text{Error in } t^2 = 0.1 \cdot 2 \cdot 2 = 10\%$$
$$\text{Total error} = \sqrt{10^2 + 5.3^2} = 11\% \rightarrow 72$$

Thus, the final answer is

$$L = (6.3 \pm 0.7) \times 10^2 \ cm$$

You may try to use the the general formula;

$$\left(\frac{\partial L}{\partial t}\right)^2 = [g(\sin\alpha - \mu\cos\alpha)\,t]^2$$

$$\left(\frac{\partial L}{\partial \mu}\right)^2 = \left[-\frac{gt^2}{2}\cos\alpha\right]^2$$

$$\left(\frac{\partial L}{\partial \alpha}\right)^2 = \left[\frac{gt^2}{2}(\cos\alpha + \mu\sin\alpha)\right]^2$$

Complete several lines of algebra and obtain the final answer.