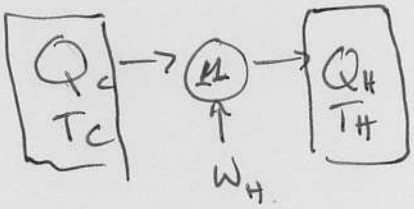
Name KEY

PH353 – Foundations of Physics II
Midterm Exam II, June 16, 2006

This is a closed book exam. However, you may use one page of notes. Please show your work. If you have questions, please do not hesitate to ask! If you need more space, extra blank sheets are available, but **please put your name on each.**

1. (25) Flora Bloom runs a Eugene business growing orchids in a greenhouse, which must be maintained at a temperature of 27° C during the winter, when the outside temperature is 7°C. She heats the greenhouse with an electrical resistance heater but her electricity bills are a whopping \$1500 per month. A customer who had taken Physics 353 suggests that it would be better to use a heat pump. What would Flora's electricity bill be with an ideal Carnot heat pump?

FOR A HEAT PUMP:



$$\boxed{Q_H = W_H + Q_C}$$

DEFINE $W_E = Q_{H,E}$ = PERFORMANCE OF ELECTRICAL RESISTANCE HEATER
= \$1500/MONTH

BUT THE HEAT PUMP DELIVERS Q_H AT COST $W_H < W_E$

TO HAVE IDEAL CARNOT EFFICIENCY $\Delta S = 0 = \frac{Q_H}{T_H} - \frac{Q_C}{T_C}$

$$\therefore \quad \frac{Q_H}{T_H} = \frac{Q_C}{T_C} \quad \text{OR} \quad \frac{Q_H}{Q_C} = \frac{T_H}{T_C}$$

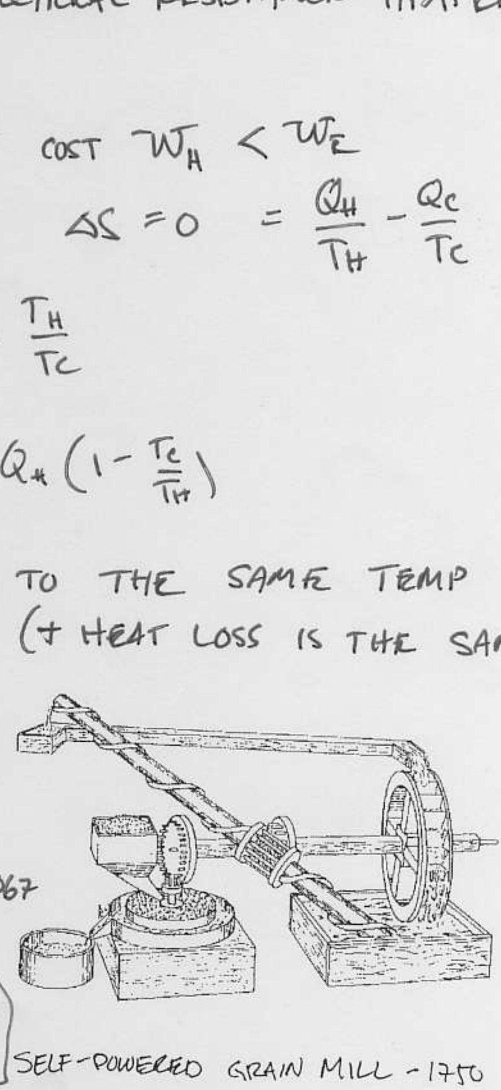
$$W_H = Q_H - Q_C = Q_H - Q_H \cdot \frac{T_C}{T_H} = Q_H\left(1 - \frac{T_C}{T_H}\right)$$

SINCE THE GREENHOUSE IS HEATED TO THE SAME TEMP REGARDLESS OF THE METHOD USED (+ HEAT LOSS IS THE SAME)
SO $Q_{H,E} = Q_H = W_E$

$$\boxed{\therefore \; W_H = W_E\left(1 - \frac{T_C}{T_H}\right)} = W_E\left(\frac{T_H - T_C}{T_H}\right)$$

$$\text{OR, } \frac{W_H}{W_E} = 1 - \frac{273+7}{273+27} = 1 - \frac{280}{300} = 0.067$$

$$\boxed{W_H = 0.067 \cdot (\$1500) = \$100 \quad \text{WOW!}}$$


SELF-POWERED GRAIN MILL - 1750

Name Key

2. A diatomic molecule in an interstellar gas cloud has a first excited rotational state energy of 4.7×10^{-4} electron volts. ($1 \text{ eV} = 1.6 \times 10^{-19}$ J)

(a) (10) Assuming an excited state degeneracy of 1, calculate the percentage of molecules in the first excited state given that the temperature of the cloud is 3 K.

(b) (15) For a diatomic molecule in rotational energy state j, the actual degeneracies of the rotational states are j(j+1), {j=0,1,2, …}. Does this make a difference for your answer in part (a)? If so, calculate the revised percentage of molecules in the first excited state.

THE QUESTION IS POORLY WORDED BUT NO ONE ASKED ABOUT IT!

- THE FRACTION OF MOLECULES IN THE 1ST EXCITED STATE ϵ_1 IS GIVEN BY $P(\epsilon_1) = \dfrac{e^{-\beta\epsilon_1}}{Z}$ BUT Z CAN'T BE CALCULATED FROM THIS INFORMATION.

- THE FRACTION OF MOLECULES IN THE 1ST EXCITED STATE ϵ_1 RELATIVE TO THE GROUND STATE IS GIVEN BY (ALWAYS)

$$\frac{P(\epsilon_1)}{P(0)} = \frac{e^{-\beta\epsilon_1}}{e^0} = e^{-\beta\epsilon_1} \quad \text{(DEGENERACY = 1)}$$

IN THIS CASE $e^{-\epsilon_1/kT} = e^{-4.7\times10^{-4}\,\text{eV}/(3)(8.62\times10^{-5}\,\text{eV/K})}$

$= 0.162$ OR 16.2%

IF THE DEGENERACY = 2 THEN $\dfrac{P(\epsilon_1)}{P(0)} = 2 \times 0.162 = 0.324$ OR 32.4%

IF YOU TRIED TO CALCULATE Z FROM $Z = 1 + e^{-\beta\epsilon_1}$ THIS WILL BE TOO SMALL BUT THE APPROACH IS NOT COMPLETELY WRONG.

Name Key

3. A model system consists of N *non-interacting* particles with just two energy levels: 0 and ε, where ε > 0. We assume that the particles are somehow weakly coupled to a heat bath at absolute temperature T.

(a) (10) Just based on the physics, how would you expect the heat capacity of the system to behave as a function of T? Briefly explain what will happen as T->0, T->∞ and at T ~ ε/k.
(b) (5) Find an expression for the partition function Z.
(c) (10) Find the mean energy of the system <E> as a function of T.
(d) (15) Find the heat capacity C(T) of the system.
(e) (15) Use the expression for the heat capacity just obtained to justify your discussion in part (a).

(a)
- WE EXPECT $C \to 0$ AS $T \to 0$ BECAUSE THERE WON'T BE ENOUGH ENERGY TO EXCITE ANY PARTICLES.
- WE EXPECT $C \to 0$ AT $T \to \infty$ BECAUSE AT HIGH T, ABOUT 50% OF PARTICLES WILL BE IN EACH STATE. THEREFORE NO MORE ENERGY CAN BE ABSORBED.
- C(T) MUST HAVE A MAXIMUM SINCE IT IS > 0, WE EXPECT FROM EXPERIENCE THAT THIS WILL BE AROUND $T \sim \epsilon/k$

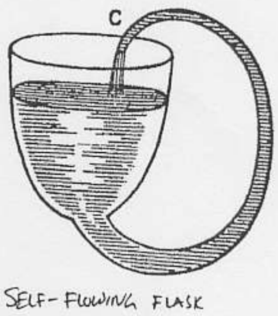
(b) $\boxed{\text{SINCE PARTICLES ARE INDEPENDENT WE CAN CALCULATE } Z, \langle E\rangle \text{ AND } C \text{ FOR ONE PARTICLE AND MULTIPLY BY } N}$

$$\frac{Z}{N} = e^{\beta\cdot 0} + e^{-\beta\epsilon} = 1 + e^{-\beta\epsilon}$$

(c) $$\frac{\langle E\rangle}{N} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{1}{1+e^{-\beta\epsilon}}\left(-\epsilon e^{-\beta\epsilon}\right) = \frac{\epsilon}{1+e^{\beta\epsilon}}$$

(d) $$\frac{C}{N} = \frac{\partial\langle E\rangle}{\partial T} = \frac{\partial\langle E\rangle}{\partial\beta}\frac{\partial\beta}{\partial T} = -\left(\frac{1}{kT^2}\right)\frac{\partial\langle E\rangle}{\partial\beta} = \frac{\epsilon^2}{kT^2}\frac{e^{\beta\epsilon}}{(1+e^{\beta\epsilon})^2}$$

$$\text{OR } \frac{C(T)}{N} = k(\beta\epsilon)^2\frac{e^{\beta\epsilon}}{(1+e^{\beta\epsilon})^2}$$



SELF-FLOWING FLASK BOYLE ~1650

(e) AS $T \to 0$ $\beta \to \infty$ $C \sim \frac{(\beta\epsilon)^2}{e^{2\beta\epsilon}} \sim 0$

AS $T \to \infty$ $\beta \to 0$ $C \to 0$

MAX AT $\partial C/\partial T = 0$ BUT THIS IS HARD! PROBLEM DOES NOT ASK FOR THE EXACT VALUE

Thanks for a great year! You've worked hard and been a good class. Good luck and have a good summer.

(SEE NEXT PAGE FOR PLOT OF C VS T)

■ Heat Capacity of two-state system, epsilon=k=1

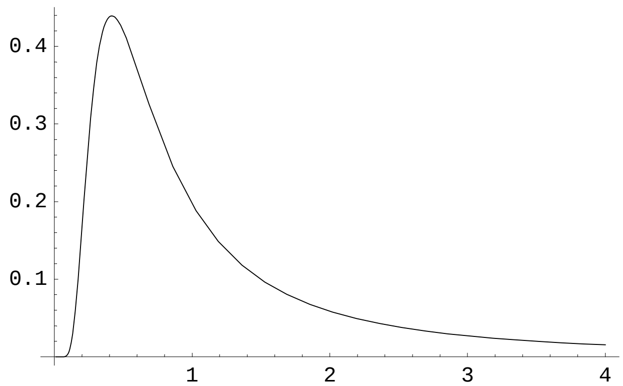
```
In[1]:= Z = 1 + Exp[-beta]
```

Out[1]= $1 + e^{-beta}$

```
In[2]:= energy = (-1 / Z) * D[Z, beta] /. beta → 1 / t;

In[3]:= heatcap = D[energy, t];

In[4]:= plot1 = Plot[heatcap, {t, 0.01, 4}]
```



Out[4]= - Graphics -

Note that the maximum is at about kT~ 0.5*epsilon, and it is surprisingly sharp! *Mathematica* has a hard time finding the maximum, though... (FindRoot is suggested)

```
In[6]:= Solve[D[heatcap, t] == 0, t]
```

Solve::tdep :
The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More...

Out[6]= $\text{Solve}\left[\frac{2\, e^{-3/t}}{(1+e^{-1/t})^3\, t^4} - \frac{3\, e^{-2/t}}{(1+e^{-1/t})^2\, t^4} + \frac{e^{-1/t}}{(1+e^{-1/t})\, t^4} + \frac{2\, e^{-2/t}}{(1+e^{-1/t})^2\, t^3} - \frac{2\, e^{-1/t}}{(1+e^{-1/t})\, t^3} == 0,\ t\right]$

```
In[8]:= FindRoot[D[heatcap, t] == 0, {t, 0.5}]
```

Out[8]= {t → 0.416778}