Physics 351
Foundations of Physics II
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1 Wave equation for waves on a string

We consider a string with tension T and mass per unit length μ. The string is stretched along the z axis. The height of the string at position z at time t is denoted $\psi(t, z)$. We assume that $|\partial\psi/\partial z| \ll 1$.

We can find the equation that ψ obeys by using $F = ma$. Consider a small piece of string between z and $z + \Delta z$. The transverse force on the left end of the string is

$$F_L = -T\,\frac{\partial\psi(t, z)}{\partial z}. \tag{1}$$

(This is $-T\tan\theta$. Actually, the force is $-T\sin\theta$. However, we are assuming that $|\theta| \ll 1$ so that $\sin\theta \approx \tan\theta$. In addition, the string has been stretched by a factor $1/\cos\theta$, so its tension should have increased. However $\cos\theta \approx 1$, so we neglect this effect.) The transverse force on the right end of the string is

$$F_R = +T\,\frac{\partial\psi(t, z + \Delta z)}{\partial z}. \tag{2}$$

Thus the net transverse force on the piece of string is

$$T\left[\frac{\partial\psi(t, z + \Delta z)}{\partial z} - \frac{\partial\psi(t, z)}{\partial z}\right] \sim T\frac{\partial^2\psi(t, z)}{\partial z^2}\,\Delta z. \tag{3}$$

we equate this to the mass $\mu\Delta z$ of the piece of string times its acceleration:

$$\mu\Delta z\,\frac{\partial^2\psi(t, z)}{\partial t^2} = T\frac{\partial^2\psi(t, z)}{\partial z^2}\,\Delta z. \tag{4}$$

We divide by Δz to obtain

$$\mu\frac{\partial^2\psi(t, z)}{\partial t^2} - T\frac{\partial^2\psi(t, z)}{\partial z^2} = 0. \tag{5}$$

If we set

$$c = \sqrt{\frac{T}{\mu}}, \tag{6}$$

then we can rewrite the equation of motion

$$\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2}\right]\psi(t,z) = 0. \tag{7}$$

This is the wave equation.

Problem 1.1 The speed of waves on a certain string is 2 m/s. Another string has a mass per unit length that is 4 times as great as the first string and a tension that is 2 times as great . What is the speed of waves on the second string.

Problem 1.2 Show that T/μ has the right units to be the square of a speed.

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2 Solution of the wave equation

It is easy to solve Eq. (7). Consider

$$\psi(t,z) = f_R(t - z/c). \tag{8}$$

Here f_R can be any smooth function we like. If

$$t - z/c = -z_0/c \tag{9}$$

denotes the argument of f at which $f(t-z/c)$ has some distinct feature, then the location of this feature along the z axis at time t is

$$z = z_0 + ct. \tag{10}$$

Thus the feature moves to the right with speed c. Now if we differentiate ψ using the chain rule, we have

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= f_R'(t - z/c) \\
\frac{\partial^2\psi}{\partial t^2} &= f_R''(t - z/c) \\
\frac{\partial\psi}{\partial z} &= -\frac{1}{c}\, f_R''(t - z/c) \\
\frac{\partial^2\psi}{\partial z^2} &= \frac{1}{c^2}\, f_R''(t - z/c).
\end{aligned} \tag{11}$$

Thus

$$\frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2}, \tag{12}$$

which is equivalent to Eq. (7).

Now consider

$$\psi(t,z) = f_L(t + z/c). \tag{13}$$

Here f_L can be any smooth function we like. If

$$t + z/c = z_0/c \tag{14}$$

denotes the argument of f at which $f(t-z/c)$ has some distinct feature, then the location of this feature along the z axis at time t is

$$z = z_0 - ct. \tag{15}$$

Thus the feature moves to the left with speed c. An analogous argument shows that this version of ψ also obeys the wave equation.

Since the wave equation is linear, the sum of two solutions is also a solution. Thus

$$\psi(t,z) = f_R(t - z/c) + f_L(t + z/c) \tag{16}$$

solves the wave equation. In fact, this is the most general solution: any solution of Eq. (7) can be written in this form. (This is proved in a course on differential equations.)

We see that disturbances move along the string with speed c – either to the left or to the right. One consequence is that if $\psi(t,z)$ is zero at time for $t < 0$ in the range $z < z_L$ and in the range $z > z_R$ then $\psi(t,z)$ for $t > 0$ will be zero for $z < z_L - ct$ and for $z > z_R + ct$. That is, ψ is zero until the disturbance gets there and the disturbance can't move faster than c.

A very important special case of this solution is

$$\psi(t,z) = A\,\cos(\omega t - kz). \tag{17}$$

Here k can be either positive or negative but must satisfy

$$|k| = \frac{\omega}{c} \tag{18}$$

so that

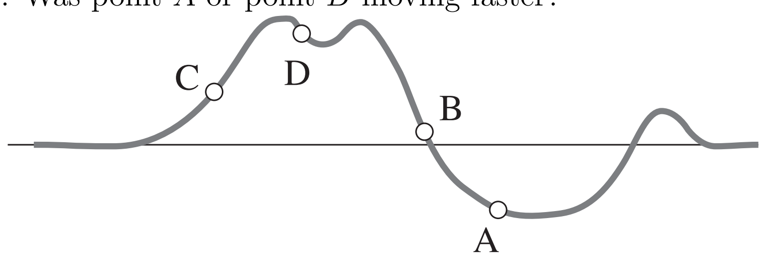
$$\psi(t,z) = A\,\cos(\omega[t \pm z/c]). \tag{19}$$

This solution represents sinusoidal waves moving either to the left or to the right with speed c. Each point on the string vibrates with period $2\pi/\omega$. The wavelength is $2\pi/|k|$.

Problem 2.1 Calculate the frequency, the wavelength and the wave speed of a wave given by

$$\psi = (0.1\ \text{cm}) \cos[(8\ \text{s}^{-1})\, t + (4\ \text{m}^{-1})\, x]. \tag{20}$$

Problem 2.2 The figure below shows a "photograph" of a string carrying a travelling wave moving from left to right. For each of the points marked, state whether the sting was moving upwards or downwards when the photograph was taken. Was point A or point B moving faster?



Problem 2.3 A certain long string has a linear density of $\mu = 0.1$ kg/m and a tension $T = 40$ N. A pulse is generated on the string by moving the left end up at a constant speed of $v_1 = 10$ m/s for a time $\tau_1 = 5$ ms, then holding the end at rest for a time $\tau_2 = 5$ ms and finally moving the end down at a constant speed of $v_3 = 10$ m/s for a time $\tau_3 = 5$ ms, thus returning the end to its original position. Draw a diagram showing the shape of the resulting right moving pulse at one particular instant. Draw another diagram showing a graph of the vertical velocity of points along the string at this instant.

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3 Reflection and transmission

Consider two strings that are joined at $z = 0$. String 1 with tension T_1 and mass per unit length μ_1 is in the region $z < 0$. String 2 with tension T_2 and mass per unit length μ_2 is in the region $z > 0$. The most realistic case is

$T_1 = T_2$, but we could have unequal tensions if the strings are tied to a ring at $z = 0$ that is free to slide along a vertical rod.

Suppose that a disturbance comes from the left along string 1 and is partially reflected at the boundary, so that

$$\psi_1(t, z) = f(t - z/c_1) + C_R f(t + z/c_1). \tag{21}$$

Here $f(t - z/c_1)$ is the incident disturbance, $C_R f(t + z/c_1)$ is the reflected disturbance, and $c_1 = \sqrt{T_1/\mu_1}$ is the speed of waves in string 1. Some of the disturbance should be transmitted into string 2, so that

$$\psi_2(t, z) = C_T f(t - z/c_2). \tag{22}$$

Here $c_2 = \sqrt{T_2/\mu_2}$ is the speed of the transmitted disturbance. We have guessed at the form of the reflected and transmitted disturbances. We will see that our guess works as long as C_R and C_T have certain values.

The strings are joined at $z = 0$, so

$$\psi_1(t, 0) = \psi_2(t, 0). \tag{23}$$

That is

$$f(t) + C_R f(t) = C_T f(t). \tag{24}$$

Thus

$$1 + C_R = C_T. \tag{25}$$

We need one more relation. The transverse force of the string 1 on string 2 is

$$\begin{aligned} F_1 &= -T_1 \frac{\partial \psi_1(t, z)}{\partial z} \\ &= -T_1 \left\{ \frac{\partial}{\partial z} [f(t - z/c_1) + C_R f(t + z/c_1)] \right\}_{z=0} \\ &= -T_1 \left[-\frac{1}{c} f'(t) + \frac{1}{c} C_R f'(t) \right] \\ &= \frac{T_1}{c_1} \left[1 - C_R \right] f'(t) \\ &= Z_1 \left[1 - C_R \right] f'(t), \end{aligned} \tag{26}$$

where

$$Z_1 = \sqrt{\mu_1 T_1} \tag{27}$$

is the *impedance* of string 1. The transverse force of string 2 on string 1 is

$$\begin{aligned} F_2 &= +T_2 \frac{\partial \psi_2(t,z)}{\partial z} \\ &= T_2 \left\{ \frac{\partial}{\partial z} C_T f(t - z/c_2) \right\}_{z=0} \\ &= -\frac{T_2}{c_2} C_T f'(t) \\ &= -Z_2 C_T f'(t), \end{aligned} \tag{28}$$

where

$$Z_2 = \sqrt{\mu_2 T_2} \tag{29}$$

is the impedance of string 2. These forces must be equal and opposite, so

$$Z_1 \Big[1 - C_R \Big] f'(t) = Z_2 C_T f'(t). \tag{30}$$

Thus

$$Z_1 \Big[1 - C_R \Big] = Z_2 C_T. \tag{31}$$

We have found two equations for the two unknowns C_R and C_T:

$$\begin{aligned} 1 + C_R &= C_T \\ Z_1 \Big[1 - C_R \Big] &= Z_2 C_T. \end{aligned} \tag{32}$$

The solution is

$$\begin{aligned} C_R &= \frac{Z_1 - Z_2}{Z_1 + Z_2} \\ C_T &= \frac{2 Z_1}{Z_1 + Z_2}. \end{aligned} \tag{33}$$

We see that there is no reflected wave and $C_T = 1$ if the impedances match. In some applications of transmission of waves across a boundary, this is a good thing.

In general there is a reflected wave. The reflection coefficient C_R is positive if $Z_1 > Z_2$ and negative if $Z_1 < Z_2$. There are two important special cases. First, suppose that string 2 is infinitely massive – for instance, you just attach string 1 to a wall. Then $Z_2 = \infty$ and $C_R = -1$. The reflected wave is the same size as the incident wave, but it is upside down. Second, suppose that string 2 is infinitely light. Then $Z_2 = 0$ and $C_R = 1$. The reflected wave is the same size as the incident wave, and it is right side up.

Problem 3.1 A certain string stretched along the z axis has a linear density of $\mu_L = 0.1$ kg/m to the left of $z = 0$ and a linear density of $\mu_R = 0.2$ kg/m to the right of $z = 0$. The string is stretched to a tension $T = 40$ N. A right moving pulse with the shape indicated in the figure is approaching the junction between the parts with different densities. I have made the drawing with the vertical scale exaggerated by a factor ten to make it easier to see. The pulse is 1 mm high and 1 cm long. The thin horizontal line is just the z axis, drawn to guide the eye. Draw what happens to the pulse at an instant after the pulse has come to the junction. The situation is a little complicated while the pulse is in the process of arriving at the junction, so make your drawing for a later time, when the situation is simple. Label distances on your drawing to make it quantitative.

Problem 3.2 A stretched string has a mass per unit length μ and a tension T. At its right end, at $z = L$, the string is tied to a massless ring that is free to slide up and down on a frictionless rod. Thus the ring provides zero vertical force, but it provides a force to the right equal to the string tension T. Now we add a "damper" attached to the right end of the string that provides a vertical force on the right end of the string equal to

$$F_y = -\gamma \, \frac{\partial \psi(t, L)}{\partial t}. \tag{34}$$

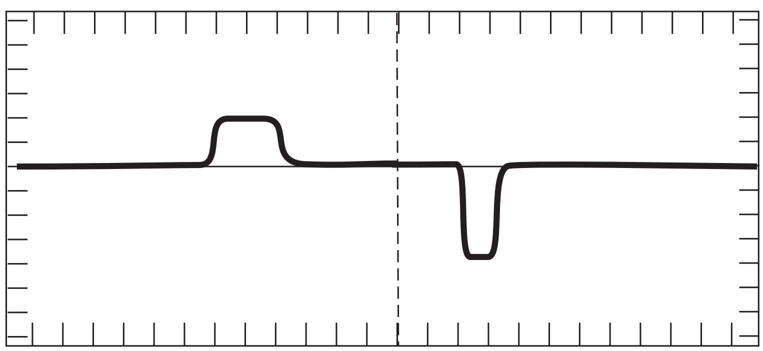
If a right moving wave comes along the string, when the wave comes to the end it will in general create a left-moving reflecting wave. Thus the total wave is

$$\psi(t, z) = f(t - z/c) + C_R \, f(t + z/c - 2L/c). \tag{35}$$

Find the reflection coefficient C_R in terms of μ, T, and γ. What value of γ will cause C_R to vanish?

Note where L goes in the reflected wave. The "$-2L/c$" term is there so that the two functions match at $z = L$: $f(t - z/c)$ becomes $f(t - L/c)$ while $f(t + z/c - 2L/c)$ becomes $f(t + L/c - 2L/c) = f(t - L/c)$. We have taken $L = 0$ so we have not seen this term before in these notes.

Problem 3.3 The mystery Pulse. A certain very long string stretched along the z axis has a constant tension but a mass per unit length that is different on the left hand part than in its right hand part. Part of the string is shown below. The boundary where the density changes is at the dashed line. In the past, the displacement of the string was zero except for a more-or-less rectangular pulse travelling from left to right. When this pulse arrived at the boundary, a reflected pulse moving to the left and a transmitted pulse travelling to the right emerged. A short time afterward, a photograph of the string was taken. The figure below shows the photograph. The pulse on the left, of height 2 units and length 2 units is the reflected pulse. The pulse on the right, of height -4 units and length 1 unit, is the transmitted pulse. Describe the original pulse, giving its height and width and saying whether its height was positive or negative.



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4 Sound waves in a solid

Consider a wave in some solid material. Our material is isotropic: it has no preferred direction. (Crystalline materials have preferred directions and are a little more complicated to analyze.) Our material obeys "Hooke's Law": it resists being squeezed or bent with a force that is proportional to how much it is squeezed or bent. We will formulate the force law more precisely below.

In the kind of wave that we consider, the material moves. Let $\{x_1, x_2, x_3\}$ be the coordinates in space and t be the time. Imagine that we put labels on points in our material. Each point is labelled by three numbers $\{R_1, R_2, R_3\}$. The components can be denoted as R_a with $a = 1, 2, 3$. The label R_a at a particular point $\vec{x}$ at a particular time t is denoted by $R_a(t, \vec{x})$. If the material is just sitting there, without any disturbance propagating through it and without being squeezed, we can choose the labels so that

$$R_a(t, \vec{x}) = x_a. \tag{36}$$

However, as soon as a disturbance comes along, $R_a(t, \vec{x})$ will be a non-trivial function of $\vec{x}$ and t.

This is the start of how to set up continuum mechanics. [See my book *Classical Field Theory*.] However, we want to deal with just the simple case of small disturbances. Thus we suppose that $R_a(t, \vec{x}) - x_a$ is small. We define

$$\psi_a(t, \vec{x}) = x_a - R_a(t, \vec{x}). \tag{37}$$

Thus $\psi_a(t, \vec{x})$ gives a description of the disturbance. We are concerned with cases in which ψ_a is small. As an example, if at some particular t and $\vec{x}$, $\psi_1(t, \vec{x}) = 0$, $\psi_2(t, \vec{x}) = 0$, and $\psi_3(t, \vec{x}) = 1.73\ \mu m$, then the material at $\vec{x}$ and t has moved $1.73\ \mu m$ in the z direction.

Let's simplify some more by considering a plane wave that moves in the z direction. That is, we suppose that $\psi_a(t, \vec{x})$ does not depend on x_1 or x_2, but it does depend on $x_3 = z$. We can also keep things simple by considering a sinusoidal wave:

$$\psi_a(t, \vec{x}) = A_a \cos(\omega t - kz). \tag{38}$$

The wave is moving in the z direction and the atoms are vibrating with amplitude A_a. The twist is that A_a has three components – it is a vector.

Longitudinal waves. Suppose that $\vec{A}$ points in the z direction: $A_1 = A_2 = 0$, but A_3 is not zero. To see what our wave will do, we need to write "$F = ma$." We consider a slab of material of thickness Δz and cross sectional area $\mathcal{A}$. The mass of the slab is $\rho \mathcal{A} \Delta z$ where ρ is the density of the substance. The z component of the force on the left hand side of the slab, at coordinate z is proportional to the area $\mathcal{A}$. The proportionality constant is something known as T_{33}. (That is, this is the definition of T_{33}, which is analogous to pressure.) Thus the force is $\mathcal{A}\, T_{33}(t, z)$. Similarly, the force on the right hand side of the slab is $-\mathcal{A}\, T_{33}(t, z + \Delta z)$. Thus the net force on the slab is

$$F_3 = -\mathcal{A}\,\frac{\partial T_{33}(t,z)}{\partial z}\,\Delta z. \tag{39}$$

Setting this equal to ma, we have

$$\rho\mathcal{A}\Delta z\frac{\partial^2\psi_3(t,z)}{\partial t^2} = -\mathcal{A}\,\frac{\partial T_{33}(t,z)}{\partial z}\,\Delta z, \tag{40}$$

or, dividing by $\mathcal{A}$ and δz,

$$\rho\frac{\partial^2\psi_3(t,z)}{\partial t^2} = -\frac{\partial T_{33}(t,z)}{\partial z}. \tag{41}$$

Now we need to know something about T_{33}. We will make a model that the material resists being squeezed or stretched. Consider T_{33} at $z = 0$. If $\partial\psi_3/\partial z = 0$ there is no internal force: $\partial\psi_3/\partial z = 0$ means that near $z = 0$ every atom has been moved the same amount, so that the material has moved a little from where it originally was, but has not been squeezed or stretched. Now suppose that the atoms to the left have been moved to the left and the atoms to the right have been moved to the right, so that $\partial\psi_3/\partial z > 0$. Then the material has been stretched. The material to the left of $z = 0$ should now be pulling the material to the right of $z = 0$ to the left: $T_{33} < 0$. On the other hand, suppose that the atoms to the left have been moved to the right and the atoms to the right have been moved to the left, so that $\partial\psi_3/\partial z < 0$. Then the material has been squeezed. The material to the left of $z = 0$ should now be pushing the material to the right of $z = 0$ to the left: $T_{33} > 0$. Thus we propose

$$T_{33} = -C_L\,\frac{\partial\psi_3}{\partial z}. \tag{42}$$

Here C_L is a coefficient characteristic of the material. If the material is stiff, C_L is big. If the material is squishy, C_L is small.

Let's put this into our $F = ma$ equation.

$$\rho\frac{\partial^2\psi_3(t,z)}{\partial t^2} = -\frac{\partial}{\partial z}\left(-C_L\,\frac{\partial\psi_3}{\partial z}\right). \tag{43}$$

That is

$$\rho\frac{\partial^2\psi_3(t,z)}{\partial t^2} - C_L\,\frac{\partial^2\psi_3}{\partial^2 z} = 0. \tag{44}$$

If we divide through by ρ, we get our wave equation in the standard form

$$\frac{\partial^2 \psi_3(t,z)}{\partial t^2} - c^2\,\frac{\partial^2 \psi_3}{\partial^2 z} = 0, \tag{45}$$

where

$$c = \sqrt{\frac{C_L}{\rho}}. \tag{46}$$

Transverse waves. Suppose that $\vec{A}$ points in the x direction: $A_2 = A_3 = 0$, but A_1 is not zero. To see what our wave will do, we need to write "$F = ma$." We consider a slab of material of thickness Δz and cross sectional area $\mathcal{A}$. The mass of the slab is $\rho\mathcal{A}\Delta z$ where ρ is the density of the substance. The 1 component of the force on the left hand side of the slab, at coordinate z is proportional to the area $\mathcal{A}$. The proportionality constant is something known as T_{13}. (That is, this is the definition of T_{13}, which is sort of a transverse pressure.) Thus the force is $\mathcal{A}\,T_{13}(t,z)$. Similarly, the force on the right hand side of the slab is $-\mathcal{A}\,T_{13}(t,z+\Delta z)$. Thus the net force on the slab is

$$F_1 = -\mathcal{A}\,\frac{\partial T_{13}(t,z)}{\partial z}\,\Delta z. \tag{47}$$

Setting this equal to ma, we have

$$\rho\mathcal{A}\Delta z\frac{\partial^2 \psi_1(t,z)}{\partial t^2} = -\mathcal{A}\,\frac{\partial T_{13}(t,z)}{\partial z}\,\Delta z. \tag{48}$$

or, dividing by $\mathcal{A}$ and δz,

$$\rho\frac{\partial^2 \psi_1(t,z)}{\partial t^2} = -\frac{\partial T_{13}(t,z)}{\partial z}. \tag{49}$$

Now we need to know something about T_{13}. We will make a model that the material resists being sheared. Consider T_{13} at $z = 0$. If $\partial\psi_1/\partial z = 0$ there is no internal force: $\partial\psi_1/\partial z = 0$ means that near $z = 0$ every atom has been moved the same amount, so that the material has moved a little from where it originally was, but has not been sheared. Now suppose that the atoms to the left have been moved to down and the atoms to the right have been moved up, so that $\partial\psi_1/\partial z > 0$. Then the material has been sheared. The material to the left of $z = 0$ should now be pulling the material to the right of $z = 0$ to the down: $T_{13} < 0$. On the other hand, suppose that the atoms to the left have been moved to the up and the atoms to the right have been moved to the down, so that $\partial\psi_1/\partial z < 0$. Then the material has been sheared in the other direction. The material to the left of $z = 0$ should now be pushing the material to the right of $z = 0$ up: $T_{13} > 0$. Thus we propose

$$T_{13} = -C_T\,\frac{\partial\psi_1}{\partial z}. \tag{50}$$

Here C_T is a coefficient characteristic of the material. If the material is stiff, C_T is big. If the material is easy to bend, C_T is small.

Let's put this into our $F = ma$ equation.

$$\rho\frac{\partial^2\psi_1(t,z)}{\partial t^2} = -\frac{\partial}{\partial z}\left(-C_T\,\frac{\partial\psi_1}{\partial z}\right). \tag{51}$$

That is

$$\rho\frac{\partial^2\psi_1(t,z)}{\partial t^2} - C_T\,\frac{\partial^2\psi_1}{\partial^2 z} = 0. \tag{52}$$

If we divide through by ρ, we get our wave equation in the standard form

$$\frac{\partial^2\psi_3(t,z)}{\partial t^2} - c^2\,\frac{\partial^2\psi_3}{\partial^2 z} = 0, \tag{53}$$

where

$$c = \sqrt{\frac{C_T}{\rho}}. \tag{54}$$

We have met two elastic constants, C_T, which applies for transverse waves, and C_L, which applies for longitudinal waves. These are not standard notations. One standard notation is to use the *Lamé* constants λ and μ, with

$$\begin{aligned} C_T &= \mu \\ C_L &= \lambda + 2\mu. \end{aligned} \tag{55}$$

Sometimes μ is called the *shear modulus* and $\lambda + 2\mu/3$ is called the *compression modulus*. Another combination that you may read about is *Young's modulus* $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$. Probably what is important is that an isotropic "Hooke's Law" solid has two elastic constants and that combinations of them control longitudinal and transverse waves. These elastic constants are quite large. For example, for copper

$$\begin{aligned} \mu &= 4.8 \times 10^{10}\ \mathrm{N/m^2} \\ \lambda &= 10 \times 10^{10}\ \mathrm{N/m^2}. \end{aligned} \tag{56}$$

The density of copper is 8.92×10^3 kg/m^3.

Problem 4.1 What is the speed of transverse waves in copper? What is the speed of longitudinal waves in copper?

Problem 4.2 A longitudinal disturbance generated by an earthquake is observed to travel 1000 km in 3 minutes. Estimate $C_L = \lambda + 2\mu$ for the rock through which the disturbance travels, assuming that the average density of the rock is 2700 kg/m^3.

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5 The stress tensor

We have made use of quantities T_{33} and T_{12}. It is about time to say what the general notation is. We think of some material that may have internal forces in it. Imagine a plane surface of area $\mathcal{A}$ oriented perpendicular to the j-axis inside the material. The material on the $-x_j$ side of this plane exerts a force $\vec{F}$ on the material that lies on the $+x_j$ side of the plane. The force is proportional to the area $\mathcal{A}$ and has components F_1, F_2, F_3. We denote $F_i/\mathcal{A}$ by T_{ij}. There are nine separate quantities T_{ij} since i can be 1,2, or 3 and so can j. Just remember that i is the direction of the force and j is the direction in which it is being transmitted. The object T_{ij} is called the *stress tensor*.

6 Waves in a fluid

A fluid (like water or air) does not resist shear. That is, $T_{13} = 0$, and more generally $T_{ij} = 0$ for $i \neq j$. Thus there are no transverse waves in a fluid. Also, $T_{11} = T_{22} = T_{33}$. Their common value is called the pressure, P.

For longitudinal waves moving in the 3-direction, we are interested in $T_{33} = P$. Recall that only $\partial T_{33}/\partial z$ entered our $F = ma$ equation for the motion of the material when a longitudinal wave comes by. In order to keep the derivation simple, I assumed that T_{33} was zero in the absence of squeezing by the wave. In fact, there could have been a constant background value of T_{33} and the wave equation would have been the same: the background pressure does not exert a net force on a little slab of material. We now imagine that there is a constant pressure P_0 in the fluid and then a little extra, ΔP, when the wave comes by. (ΔP can have either sign.) In our derivation, we made the assumption that

$$\Delta P = -C_L \frac{\partial \psi_3}{\partial z} \tag{57}$$

for some constant C_L that is characteristic of the material. This works for fluids. However, C_L is not the standard notation, so let's relate C_L to the quantities people usually use to talk about fluids.

In a fluid, the pressure is a function $P(\rho)$ of the density ρ. Let the density of the undisturbed fluid be ρ_0 and the pressure in the undisturbed fluid be $P_0 = P(\rho_0)$. If we squeeze the fluid in the 3-direction, the new density is

$$\rho = \rho_0 \times \frac{\partial R_3}{\partial z}. \tag{58}$$

(A slab of fluid in a box of size $\Delta R_1 \times \Delta R_2 \times \Delta R_3$ is now squeezed into a box of the same size in the 1- and 2-directions, but size Δz in the 3-direction.) Since

$$R_3 = z - \psi_3(t, z), \tag{59}$$

we have

$$\rho = \rho_0 \times \left(1 - \frac{\partial \psi_3}{\partial z}\right). \tag{60}$$

Thus

$$\Delta \rho = -\rho_0 \frac{\partial \psi_3}{\partial z}. \tag{61}$$

Then

$$\Delta P = \frac{dP(\rho_0)}{d\rho} \, \Delta \rho = -\rho_0 \, \frac{dP(\rho_0)}{d\rho} \, \frac{\partial \psi_3}{\partial z}. \tag{62}$$

The quantity $1/[\rho \, dP(\rho)/d\rho]$ is called the *compressibility* κ of the fluid:

$$\kappa = \frac{1}{\rho} \frac{d\rho}{dP}. \tag{63}$$

(In writing this, we think of ρ being a function of P instead of P being a function of ρ. At some risk of being to obvious, I note that

$$\frac{dP}{d\rho} = \frac{1}{\frac{d\rho}{dP}}. \tag{64}$$

The compressibility is a function of P, but what we want is the compressibility at the standard pressure P_0. Then ρ in Eq. (63) is really ρ_0.) The compressibility is big if an increase in pressure causes a big increase in density. Air is pretty compressible (big κ) while water is pretty incompressible (small κ). In terms of κ, we have

$$\Delta P = -\frac{1}{\kappa}\frac{\partial \psi_3}{\partial z}. \tag{65}$$

Comparing to Eq. (57), we have

$$C_L = \frac{1}{\kappa}. \tag{66}$$

This gives the speed of sound in a fluid:

$$c = \sqrt{\frac{1}{\kappa\rho}}. \tag{67}$$

Problem 6.1 The compressibility of water is $\kappa_w = 4.9 \times 10^{-10}\ \mathrm{m^2\,N^{-1}}$. The compressibility of air is $\kappa_a = 7.1 \times 10^{-6}\ \mathrm{m^2\,N^{-1}}$. The density of water is $\rho_w = 1.0 \times 10^3\ \mathrm{kg\,m^{-3}}$. The density of air is $\rho_a = 1.1\ \mathrm{kg\,m^{-3}}$. (All of these numbers are approximate values for room temperature and atmospheric pressure). Find the speed of sound in water and the speed of sound in air. Suppose we had some kryptonite, a fictional foul smelling green liquid that has the density of water but the compressibility of air. What would the speed of sound in kryptonite be?

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7 Impedance in a solid or fluid

We studied how waves on a string are reflected and transmitted at a boundary between two kinds of string. The same analysis applies to transverse and longitudinal waves in a solid or fluid. Let's say that there is a wave moving in the 3-direction in material I and comes to a boundary at $z = 0$ where it enters material II. Just to be definite, let's consider a longitudinal wave.

1) The motion of the material must be continuous at the boundary:

$$\psi_3^{(I)}(t,0) = \psi_3^{(II)}(t,0). \tag{68}$$

2) The force has to match at the boundary:

$$T_{33}^{(I)}(t,0) = T_{33}^{(II)}(t,0). \tag{69}$$

That is

$$-C_L^{(I)}\,\frac{\partial\psi_3^{(I)}(t,0)}{\partial z} = -C_L^{(II)}\,\frac{\partial\psi_3^{(II)}(t,0)}{\partial z}. \tag{70}$$

We suppose that there is an incident wave $f(t-z/c_I)$ and a reflected wave $Rf(t+z/c_I)$ in material I and a transmitted wave $Tf(t-c/c_{II})$ in material II. (I had to change notation because, alas, we are using C_T for the elastic modulus for transverse waves so it can't be the transmission coefficient). Then condition 1 gives

$$1 + R = T. \tag{71}$$

Condition 2 gives

$$-C_L^{(I)}\left(-\frac{1}{c_I}\frac{df(t)}{dt} + \frac{R}{c_I}\frac{df(t)}{dt}\right) = -C_L^{(II)}\left(-\frac{T}{c_{II}}\frac{df(t)}{dt}\right). \tag{72}$$

That is

$$\frac{C_L^{(I)}}{c_I}\,(1-R) = \frac{C_L^{(II)}}{c_{II}}\,T. \tag{73}$$

Since

$$c = \sqrt{\frac{C_L}{\rho}}, \tag{74}$$

it is useful to define the impedance for longitudinal waves

$$Z_L = \sqrt{\rho C_L}. \tag{75}$$

Then

$$Z_L^{(I)}\,(1-R) = Z_L^{(II)}\,T. \tag{76}$$

Solving Eqs. (71), (76), we obtain

$$\begin{aligned} R &= \frac{Z_L^{(I)} - Z_L^{(II)}}{Z_L^{(I)} + Z_L^{(II)}}, \\ T &= \frac{2Z_L^{(I)}}{Z_L^{(I)} + Z_L^{(II)}}. \end{aligned} \tag{77}$$

The same result applies to transverse waves, with

$$Z_T = \sqrt{\rho C_T}. \tag{78}$$

Also, if we are talking about longitudinal waves in a fluid, then we can use $C_L = 1/\kappa$, so

$$Z_L = \sqrt{\frac{\rho}{\kappa}}. \tag{79}$$

Problem 7.1 The compressibility of water is $\kappa_w = 4.9 \times 10^{-10}\ \mathrm{m^2\,N^{-1}}$. The compressibility of air is $\kappa_a = 7.1 \times 10^{-6}\ \mathrm{m^2\,N^{-1}}$. The density of water is $\rho_w = 1.0 \times 10^3\ \mathrm{kg\,m^{-3}}$. The density of air is $\rho_a = 1.1\ \mathrm{kg\,m^{-3}}$. (All of these numbers are approximate values for room temperature and atmospheric pressure). What is the reflection coefficient for a sound wave going from air and coming to the surface of some water? (Assume that the wave direction is perpendicular to the water surface).

Problem 7.2 What is the reflection coefficient for a sound wave going from water and coming to the surface of some copper? (Assume that the wave direction is perpendicular to the surface. Recall that sound waves in a fluid are longitudinal).

---

8 Energy in waves

Waves transmit energy. It's pretty simple.

String The string to the left of some point z exerts a vertical force

$$f = -Zc\,\frac{\partial\psi}{\partial z} \tag{80}$$

on the string to the right, where $Zc = T$. This relation is the definition of the impedance Z. The wave thus transmits power

$$P = f\,\frac{\partial\psi}{\partial t} \tag{81}$$

to the string on the right. Thus we can write the power as

$$P = -Z\,c\,\frac{\partial\psi}{\partial z}\,\frac{\partial\psi}{\partial t}. \tag{82}$$

Transverse waves. The material to the left of some point z exerts a transverse force per unit area $f \equiv T_{13}$

$$f = -Zc\,\frac{\partial\psi_1}{\partial z} \tag{83}$$

on the material to the right, where $Zc = C_T$. It thus transmits power per unit area

$$P/\mathcal{A} = f\,\frac{\partial\psi_1}{\partial t} \tag{84}$$

to the material on the right. We can write this as

$$P/\mathcal{A} = -Z\,c\,\frac{\partial\psi_1}{\partial z}\,\frac{\partial\psi_1}{\partial t}. \tag{85}$$

Longitudinal waves. The material to the left of some point z exerts a longitudinal force per unit area $f = T_{33}$

$$f = -Zc\,\frac{\partial\psi_3}{\partial z} \tag{86}$$

on the material to the right, where $Zc = C_L$. (In a fluid, $f = T_{33} - T_{33}^0$ is the change in the pressure above the ambient pressure. One would normally call it p or ΔP, but f fits with our standard notation.)

It thus transmits power per unit area

$$P/\mathcal{A} = f\,\frac{\partial\psi_3}{\partial t} \tag{87}$$

to the material on the right. We can write this as

$$P/\mathcal{A} = -Z\,c\,\frac{\partial\psi_3}{\partial z}\,\frac{\partial\psi_3}{\partial t}. \tag{88}$$

We see that the same formula applies in each case, except that we have power per unit area for continuous media and just power for waves on a string.

Let's see what this is in the case of a sinusoidal vibration wave (either transverse or longitudinal) moving to the right in some material

$$\psi(t,z) = A\,\cos\Big(\omega(t - z/c)\Big). \tag{89}$$

We get

$$
\begin{aligned}
P/\mathcal{A} &= -Z\,c\left(\frac{A\omega}{c}\right)(-A\omega)\sin^2\!\Big(\omega(t-z/c)\Big) \\
&= -Z\,A^2\,\omega^2\sin^2\!\Big(\omega(t-z/c)\Big). \qquad (90)
\end{aligned}
$$

The time average of the power per unit area is then

$$
\langle P\rangle/\mathcal{A} = \tfrac{1}{2}\,ZA^2\,\omega^2 \qquad (91)
$$

since the time average of a squared sine or cosine function is 1/2.

The time average of the power per unit area in a sinusoidal wave is often called the *intensity* I of the wave. Thus, the intensity is

$$
I = \tfrac{1}{2}\,ZA^2\,\omega^2 \qquad (92)
$$

This is pretty general: the power is proportional to the *square* of the amplitude and also to ω^2. There is also a factor of the impedance.

Let's try this for reflection and transmission of a wave. Suppose that medium I lies in $z < 0$ and medium II lies in $z > 0$. Let there be a wave coming from the left in medium I, together with a reflected wave

$$
\psi_I(t,z) = A\,\cos\!\Big(\omega(t-z/c_I)\Big) + R\,A\,\cos\!\Big(\omega(t+z/c_I)\Big). \qquad (93)
$$

In medium II there is a transmitted wave

$$
\psi_{II}(t,z) = T\,A\,\cos\!\Big(\omega(t-z/c_{II})\Big). \qquad (94)
$$

The average power in the transmitted wave is

$$
I_{II} = \tfrac{1}{2}\,Z_{II}T^2\,A^2\,\omega_{II}^2. \qquad (95)
$$

It's a little trickier to work out the power in the medium I:

$$
\begin{aligned}
P/\mathcal{A} &= -Z_I\,c_I\left[\frac{A\omega}{c_I}\,\sin\!\Big(\omega(t-z/c_I)\Big) - \frac{RA\omega}{c_I}\,\sin\!\Big(\omega(t+z/c_I)\Big)\right] \\
&\quad\times\Big[-A\omega\,\sin\!\Big(\omega(t-z/c_I)\Big) - RA\omega\,\sin\!\Big(\omega(t+z/c_I)\Big)\Big] \\
&= Z_IA^2\omega^2\left[\sin^2\!\Big(\omega(t-z/c_I)\Big) - R^2\,\sin^2\!\Big(\omega(t+z/c_I)\Big)\right] \qquad (96)
\end{aligned}
$$

The time average of this is

$$
\langle P_I\rangle/\mathcal{A} = \tfrac{1}{2}\,Z_IA^2\omega^2\,[1-R^2]. \qquad (97)
$$

This makes great sense. First, there is the power that would be transmitted by the incident wave by itself. Then there is the power that would be transmitted by the reflected wave by itself, with A replaced by RA. But the power transmitted by the reflected wave comes with a minus sign because the reflected wave is moving in the $-z$ direction. The total power transmitted to the right is just the sum of these:

$$\langle P_I \rangle / \mathcal{A} = I_{\rm inc} - I_{\rm refl}. \tag{98}$$

Is energy conserved? We should have

$$\langle P_I \rangle / \mathcal{A} = \langle P_{II} \rangle / \mathcal{A}. \tag{99}$$

That is, we need

$$Z_I [1 - R^2] = Z_{II} T^2 \tag{100}$$

Since $T = 1 + R$, the needed relation is

$$Z_I [1 - R][1 + R] = Z_{II} [1 + R]^2 \tag{101}$$

or

$$Z_I [1 - R] = Z_{II} [1 + R] \tag{102}$$

If we solve this for R we get

$$R = \frac{Z_I - Z_{II}}{Z_I + Z_{II}}. \tag{103}$$

This is just the relation we found earlier for R.

There is a simple lesson. A fraction R^2 of the incident energy is reflected. The rest, a fraction $1 - R^2$, is transmitted.

Note that, in the case that R is close to 1, the transmission coefficient T is not small, $T \approx 2$. But the power in the transmitted wave is small: $(1+R)(1-R) \approx 2(1-R)$. The explanation is that R is close to 1 when Z_{II} is small compared to Z_I, so that the power $(1/2)\, Z_{II} T^2 \, A^2 \, \omega_{II}^2$ is small. This happens, for instance, when a sound wave moves from water into air.

Problem 8.1 A transverse sound wave travels along the z-axis in an isotropic solid, *medium I*. This solid joins a second isotropic solid, *medium II*, at $z = 0$. The characteristics of the two solids are such that one quarter of the energy in the incident wave is reflected and three quarters is transmitted into medium II. What can you conclude about the reflection and transmission coefficients R and T? What can you conclude about the ratio Z_I / Z_{II} of the impedances of the two materials? (Be careful, there are two solutions.)
