Physics 351 Foundations of Physics II Notes on Oscillators Davison E. Soper

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## 1 The simple harmonic oscillator

We begin with Newton's second law of motion, "F = ma," written as

$$m\ddot{\psi} = F \tag{1}$$

where  $\psi$  represents the position of a particle with mass m and F is the force acting on the particle. For the simple harmonic oscillator, the force is

$$F = -s\psi, \tag{2}$$

where s represents, for instance, the stiffness of a spring that acts to restore the particle to position  $\psi = 0$ . Thus

$$\ddot{\psi} = -\omega_0^2 \psi, \tag{3}$$

where

$$\omega_0 = \sqrt{\frac{s}{m}}.\tag{4}$$

[The variable  $\psi$  is often called x and s is often called k.]

### 2 Solution

A solution of Eq. (3) is

$$\psi(t) = A \cos(\omega_0 t + \phi). \tag{5}$$

where A and  $\phi$  are constants. To verify this, differentiate (using the chain rule):

$$\psi(t) = -A\omega_0 \sin(\omega_0 t + \phi) \tag{6}$$

 $\mathbf{SO}$ 

$$\ddot{\psi}(t) = -A\omega_0^2 \cos(\omega_0 t + \phi). \tag{7}$$

That is  $\ddot{\psi} = -\omega_0^2 \psi$ .

The essential feature of this solution is that it oscillates. The "angular frequency" of the oscillation is  $\omega_0$ . The particle returns to where it was after a time  $\tau$ , called the "period," given by  $\omega_0 \tau = 2\pi$ , or

$$\tau = \frac{2\pi}{\omega_0}.\tag{8}$$

The "frequency" of the oscillation is defined to be  $\nu_0 = 1/\tau$ , or

$$\nu_0 = \frac{\omega_0}{2\pi}.\tag{9}$$

The quantity A is the "amplitude" of the oscillation: the maximum value of  $\psi$ . The quantity  $\phi$  is the "phase constant."

Problem 2.1 If the oscillator described by Eqs. (1) and (2) has a mass m = 0.01 kg and a spring constant s = 16 N/m, calculate the frequency, the angular frequency, and the period of the oscillations.

## 3 Initial conditions

There is a solution for any initial position  $\psi(0)$  and velocity  $\dot{\psi}(0)$ . If we are given  $\{\psi(0), \dot{\psi}(0)\}$ , we can adjust A and  $\phi$  to match:

$$\psi(0) = A\cos(\phi)$$
  

$$\dot{\psi}(0) = -A\omega_0\sin(\phi).$$
(10)

It is usually simplest to write down Eq. (10) with the initial conditions you have in mind and then solve the equation for A and  $\phi$ . However, let us verify that we can solve Eq. (10) no matter what the initial conditions are. The solution is

$$A = \left[\psi(0)^2 + (\dot{\psi}(0)/\omega_0)^2\right]^{1/2} \tag{11}$$

and

$$\tan(\phi) = -\frac{\psi(0)}{\omega_0 \psi(0)}.$$
 (12)

Since  $\tan(\phi + \pi) = \tan(\phi)$ , there are two solutions of Eq. (12). You can determine which solution you want by determining the sign of  $\cos(\phi)$  or  $\sin(\phi)$  from Eq. (10).

Problem  $\overline{3.1}$  Suppose that  $\psi(t)$  obeys Eq. (3) with  $\omega_0 = 1 \text{ s}^{-1}$ . The system is started from  $\psi = 0$  with an initial velocity  $\dot{\psi} = 0.2 \text{ m/s}$ . What are the amplitude and phase in Eq. (5) for the solution?

Problem 3.2 Suppose that  $\psi(t)$  obeys Eq. (3) with  $\omega_0 = 0.3 \text{ s}^{-1}$ . The system is started from rest with an initial position  $\psi = -1.2$  m. What are the amplitude and phase in Eq. (5) for the solution?

### 4 Vector diagrams

It is sometimes useful to represent the solution  $\psi(t) = A \cos(\omega_0 t + \phi)$  by means of a diagram in which we use two coordinates,

$$\psi(t) = A \cos(\omega_0 t + \phi)$$
  

$$\eta(t) = A \sin(\omega_0 t + \phi).$$
(13)

We can consider  $\{\psi, \eta\}$  as a point in a plane. Bringing in another coordinate might seem to just make our lives more complicated, but it is nice because  $\{\psi, \eta\}$  lies on a circle of radius A and moves around the circle with uniform angular velocity  $\omega_0$ .

If there are two oscillators with different values of A and  $\phi$ , drawing a vector diagram can help us see the relationship between the two solutions.

We can also use a vector diagram to picture the velocity and acceleration. For the velocity, we have

$$\dot{\psi}(t) = -\omega_0 A \sin(\omega_0 t + \phi) = A \cos(\omega_0 t + \phi + \pi/2)$$
  
$$\dot{\eta}(t) = \omega_0 A \cos(\omega_0 t + \phi) = A \sin(\omega_0 t + \phi + \pi/2).$$
(14)

For the acceleration, we have

$$\ddot{\psi}(t) = -\omega_0^2 A \cos(\omega_0 t + \phi) = \omega_0^2 A \cos(\omega_0 t + \phi + \pi) \ddot{\eta}(t) = -\omega_0^2 A \sin(\omega_0 t + \phi) = \omega_0^2 A \sin(\omega_0 t + \phi + \pi).$$
(15)

Each of these vectors moves in a circle with angular velocity  $\omega_0$ , but the velocity vector moves with a phase advance of  $\pi/2$  and the acceleration moves with a phase advance of  $\pi$ .

Thus even though " $\eta$ " is just something we made up, drawing the solution this way can help us to picture what is going on.

Later, we will use complex numbers to represent our oscillator solutions. Then we will see that  $\psi$  and  $\eta$  here are just the real and imaginary parts of the solution.

## 5 Energy

Our mass has a kinetic energy

$$T = \frac{1}{2} m \dot{\psi}^2. \tag{16}$$

The spring has potential energy

$$V = \frac{1}{2} s\psi^2. \tag{17}$$

The total energy is

$$W = \frac{1}{2}m\dot{\psi}^2 + \frac{1}{2}s\psi^2.$$
 (18)

Energy conservation tells us that W should be conserved: dW/dt = 0.

We can check this directly from the differential equation:

$$dW/dt = m\ddot{\psi}\dot{\psi} + s\dot{\psi}\psi$$
  
=  $m(-\frac{s}{m}\psi)\dot{\psi} + s\dot{\psi}\psi$   
= 0. (19)

The kinetic energy and the potential energy each change with time, but their sum is constant.

Problem 5.1 Suppose the oscillator described by Eqs. (1) and (2) has a mass m = 0.01 kg and a spring constant s = 16 N/m. It is set into oscillation with an amplitude A = 0.01 m and a phase constant  $\phi = 0.367$  in Eq. (5). What is its energy W? If we change the phase angle to  $\phi = 1.263$ , what is the energy?

### 6 Complex numbers

Complex numbers and functions of complex numbers are very useful in physics.

A complex number z is a pair of real numbers  $\{x, y\}$ . One calls x the real part of z and y the imaginary part of z: x = Re z, y = Im z.

We can add two complex numbers according to the rule

$$\{x_1, y_1\} + \{x_2, y_2\} = \{x_1 + x_2, y_1 + y_2\}.$$
 (20)

We can subtract two complex numbers according to the rule

$$\{x_1, y_1\} - \{x_2, y_2\} = \{x_1 - x_2, y_1 - y_2\}.$$
 (21)

We can multiply two complex numbers according to the rule

$$\{x_1, y_1\} \times \{x_2, y_2\} = \{x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1\}.$$
 (22)

We can divide two complex numbers according to the rule

$$\frac{\{x_1, y_1\}}{\{x_2, y_2\}} = \left\{\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{-x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}\right\}.$$
(23)

There is a complex number "zero,"

$$0 = \{0, 0\} \tag{24}$$

such that

$$z + 0 = z. \tag{25}$$

There is also a complex number "one,"

$$1 = \{1, 0\} \tag{26}$$

such that

$$z \times 1 = z. \tag{27}$$

We think of the familiar real numbers as being a subset of the complex numbers by identifying a complex number of the form  $\{x, 0\}$  with the real number x. For complex numbers  $\{x_1, 0\}$ ,  $\{x_2, 0\}$ , our rules for adding, subtracting, multiplying, and dividing all reduce to the usual rules for these operations when we identify  $\{x, 0\} \to x$ . Similarly, the complex numbers 0 and 1 are identified with the real numbers 0 and 1 under the identification  $\{x, 0\} \rightarrow x$ .

With these definitions, all of the usual rules of algebra work. For instance, if a, b, and c are complex numbers with

$$\frac{a}{b} = c. \tag{28}$$

Then

$$a = b c. (29)$$

This is not obvious. You have to prove it. Similarly, one should systematically write down all of the rules of algebra and prove, one by one, that they are correct.

Working with complex numbers is easier than it might seem from the mathematical definitions given above. You need one more bit of knowledge: there is a complex number i with the property

$$i \times i = -1. \tag{30}$$

The complex number i is  $i = \{0, 1\}$ . Then the number  $z = \{x, y\}$  is z = x+iy. It is usually convenient to write x + iy instead of  $\{x, y\}$ . Now we can just use the ordinary rules of arithmetic to multiply and divide. For multiplication, we have

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2).$$
(31)

For division, we have

$$\frac{(x_1 + iy_1)}{(x_2 + iy_2)} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\
= \frac{x_1x_2 - i^2y_1y_2 + iy_1x_2 - ix_1y_2}{x_2^2 - i^2y_2^2 + ix_2y_2 - ix_2y_2} \\
= \frac{x_1x_2 + y_1y_2 + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}.$$
(32)

One sometimes uses the "complex conjugate"  $z^*$  of a number z = x + iy(with x and y real). The complex conjugate is simply  $z^* = x - iy$ . In fact, we used the complex conjugate of  $(x_2 + iy_2)$  in the derivation just given.

#### **Problem 6.2** What are the solutions of $z^2 + 2z + 2 = 0$ ?

One can have functions f of a complex variable, such that for every complex number z, f(z) is a complex number. The study of functions of a complex variable is a big area of mathematics. For our purposes, we will use polynomial functions and ratios of polynomials and one more function, the exponential function

$$\exp(z) = e^z. \tag{33}$$

We can adopt the definition (for x and y real numbers)

$$\exp(x+iy) = \exp(x)[\cos(y)+i\sin(y)], \qquad (34)$$

which relates  $\exp(z)$  to functions of real variables that we already know. If we take this as the definition, then we can prove (with a little work) that

$$\exp(z_1 + z_2) = \exp(z_1) \, \exp(z_2). \tag{35}$$

A particular example of Eq. (34) is

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta). \tag{36}$$

We will use the exponential function in studying oscillators.

Problem 6.3 How would you write  $\exp(i\pi)$  in the form x + iy?

Problem 6.4 How would you write  $\exp(i\pi/2)$  in the form x + iy?

Problem 6.5 How would you write  $i^i$  in the form x + iy?

Given a complex number z-x+iy, one defines the "complex conjugate"  $z^{\ast}$  by

$$z^* = x - iy. \tag{37}$$

We define the "absolute value" of z by

$$|z| = \sqrt{x^2 + y^2}.$$
 (38)

Note that

$$z^*z = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2.$$
 (39)

Thus

$$z^* z = |z|^2. (40)$$

We use |z| to measure the "size" of a complex number z. In physics talk, we say that z = x + iy is small if both x and y are small, which is the same thing as saying that |z| is small. In math talk, we say that a sequence  $z_j = x_j + iy_j$  approaches 0 if the sequence of real numbers  $|z_i|$  approaches zero.

We have used z = x + iy to represent a complex number z. We can also use

$$z = A e^{i\phi} \tag{41}$$

where A and  $\phi$  are real numbers and  $A \ge 0$ . We call A the amplitude and  $\phi$  the phase of z. Then

$$z = A\cos\phi + iA\sin\phi \tag{42}$$

so we can find the real and imaginary parts of z if we know the amplitude and phase, and vice versa.

Note that with this definition of A and  $\phi$ ,

$$|z| = A \tag{43}$$

and

$$|e^{i\phi}| = 1. \tag{44}$$

One can do calculus with complex numbers. The definition of derivative is

$$f'(z) \equiv \frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$
(45)

The limit has to exist for  $\Delta z \to 0$  in any direction. That is,  $\Delta z$  could be purely real or purely imaginary or something in between. A more physics minded way to state this is that

$$f(z + \Delta z) - f(z) = f'(z) \Delta z + R(\Delta z)$$
(46)

where  $R(\Delta z)$  is a function such that  $R(\Delta z)/\Delta z \to 0$  zero as  $\Delta z \to 0$ . If we write  $\Delta z = \Delta x + i\Delta y$ , then

$$f(z + \Delta z) - f(z) = f'(z) \,\Delta x + i f'(z) \,\Delta x + R(\Delta z) \tag{47}$$

The crucial requirement is that the same complex number f'(z) multiplies both  $\Delta x$  and  $i\Delta y$ . Most functions of x and y that someone might make up do not have this property. But when a complex function of a complex variable z does have this property, we say that it is differentiable and its derivative is f'(z). If f'(z) exists in a region of the complex plane, the function is said to be "analytic" in that region. Analytic functions have several wonderful properties, which are explored in a course on complex analysis.

For our purposes we need to know just a few facts. To start with, there are general rules for calculus

$$\frac{d}{dz}(af(z) + bg(z)) = af'(z) + bg'(z),$$
(48)

$$\frac{d}{dz}(f(z)g(z)) = f'(z)g(z) + f(z)g'(z),$$
(49)

and the chain rule

$$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z).$$
(50)

Next, there are just a couple of kinds of functions that we will encounter. First, all polynomials are analytic functions. If

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_N z^N$$
(51)

then

$$f'(z) = c_1 + 2c_2 z + \dots + Nc_N z^{N-1}$$
(52)

Second,  $\exp(z)$  is an analytic function. Its derivative is

$$\frac{d}{dz}\exp(z) = \exp(z). \tag{53}$$

In particular, this means that

$$\frac{d}{dt}\exp(at+b) = a\exp(at+b) \tag{54}$$

for any complex numbers a and b and a real variable t. (This holds also for a complex variable t too, but I have in mind applications in which t is the time.) To derive this, let z(t) = at + b, Then use the "chain rule"

$$\frac{d}{dt}\exp(z(t)) = \frac{dz(t)}{dt} \times \frac{d}{dz}\exp(z) = a \times \exp z.$$
(55)

There is a lot of material in this section! On the other hand, it is almost all stuff that you know already. The crucial fact is that what you have already learned about algebra and calculus applies also for complex numbers and analytic complex functions of complex numbers. Of course, one really needs to prove everything here. I don't really recommend doing that, but it is good to prove a few of these properties for yourself just to see how it works.

## 7 Different forms of the solution

There are (at least) four ways to represent our solution for the harmonic oscillator:

$$\begin{aligned}
\psi(t) &= A\cos(\omega_0 t + \phi), \\
\psi(t) &= B_p \cos(\omega_0 t) + B_q \sin(\omega_0 t), \\
\psi(t) &= C \exp(i\omega_0 t) + C^* \exp(-i\omega_0 t), \\
\psi(t) &= \operatorname{Re}\{D \exp(i\omega_0 t)\}.
\end{aligned}$$
(56)

In the last two equations, we have used complex numbers, as discussed above. We use the relation  $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ . To show that the last form is equivalent to the first form, we let  $D = A \exp(i\phi)$  and write

$$\operatorname{Re}\{D\exp(i\omega_{0}t)\} = \operatorname{Re}\{A\exp(i\phi)\exp(i\omega_{0}t)\}\$$

$$= \operatorname{Re}\{A\exp(i\omega_{0}t + i\phi)\}\$$

$$= \operatorname{Re}\{A\exp(i[\omega_{0}t + \phi])\}\$$

$$= \operatorname{Re}\{A[\cos(\omega_{0}t + \phi) + i\sin(\omega_{0}t + \phi)]\}\$$

$$= A\cos(\omega_{0}t + \phi). \tag{57}$$

With this preparation, we can look into examples of oscillators in physics, beyond just the mass on a spring.

Problem 7.1 Suppose the oscillator described by Eqs. (1) and (2) has a mass m = 2.0 kg and a spring constant s = 8 N/m. It is set into oscillation with an amplitude A = 0.1 m and a phase constant  $\phi = \pi/2$  in the first of Eqs. (??). Express this same motion in the form of the second, third, and fourth equation in Eqs. (??), with numerical values for the parameters  $\omega_0$ ,

 $B_p$ ,  $B_q$ , C, and D. When writing the complex parameters C, and D, write them in the form  $|C| \exp(i\phi_C)$  and  $|D| \exp(i\phi_D)$ .

Problem 7.2 Calculate the maximum acceleration, in units of the acceleration of gravity g, of a pickup stylus on a record player that is reproducing some music with a frequency of 2 kHz with an amplitude of 0.01 mm. (You may need to go to an antique store to find a record player to examine.)

## 8 Mass hanging on a spring

Imagine a body with mass m hanging from the end of a spring of spring constant s, the other end of which is fixed. The force is

$$F = -sx - mg \tag{58}$$

where x is the vertical position of the body. The force vanishes at position  $x_0 = -mg/s$ . Let  $\psi = x - x_0$ . We can rewrite the force as

$$F = -s\psi. \tag{59}$$

Then the equation of motion is exactly the harmonic oscillator equation of motion,

$$m\tilde{\psi} = -s\psi. \tag{60}$$

*Problem 8.1* An astronaut on the surface of the moon weighs rock samples using a light spring balance. The balance, which was calibrated on earth, has a scale 100 mm long which reads from 0 to 1 kg. The astronaut places a certain rock on the balance. The spring oscillates, with a period of 1.0 s. The astronaut waits for friction to bring the system to equilibrium. Then the balance reads 0.4 kg. What is the acceleration due to gravity on the moon? There was a tiny amount of friction, but you can ignore this friction in solving the problem.

### 9 Torsional vibrations

Another example of an oscillator is a torsional vibrator. We imagine an object that can rotate about an axis and we let  $\psi$  be the angle through which it has rotated from its equilibrium position. Then

$$I\ddot{\psi} = T_s \tag{61}$$

where  $T_s$  is the torque on the object and I is the moment of inertia of the object. (For example,  $I = \frac{1}{2}MR^2$  for a disk.) We get a harmonic oscillator if

$$T_s = -c\psi \tag{62}$$

where c is a constant. Then

$$\ddot{\psi} = -\omega_0^2 \psi \tag{63}$$

where

$$\omega_0 = \sqrt{\frac{c}{I}}.\tag{64}$$

Thus the oscillator will oscillate with angular frequency  $\omega_0$ :

$$\psi = A\cos(\omega_0 t + \phi). \tag{65}$$

#### 10 The simple pendulum

Consider a ball of mass m suspended from a fixed point by means of a string of length l. Let the angle that the string makes with the vertical direction be  $\psi$ . Then

$$I\ddot{\psi} = T_s \tag{66}$$

where the moment of inertia of the ball about the suspension point is simply  $I = ml^2$ . The torque  $T_s$  is from the force of gravity. The string exerts a force on the ball, but this force is directed toward the point of suspension so it produces no torque. The torque produced by gravity is

$$T_s = -mgl\sin(\psi). \tag{67}$$

Thus

$$\ddot{\psi} = -\frac{mgl}{ml^2}\sin(\psi) \tag{68}$$

or

$$\ddot{\psi} = -\frac{g}{l}\sin(\psi). \tag{69}$$

This is not exactly the harmonic oscillator equation. But suppose that  $\psi$  is small. Then

$$\sin(\psi) \approx \psi \tag{70}$$

so that the equation becomes

$$\ddot{\psi} = -\frac{g}{l}\psi. \tag{71}$$

This is

$$\ddot{\psi} = -\omega_0^2 \psi. \tag{72}$$

where

$$\omega_0 = \sqrt{\frac{g}{l}}.\tag{73}$$

Again the solutions are oscillatory:

$$\psi = A\cos(\omega_0 t + \phi). \tag{74}$$

The angular frequency gets smaller as the length l becomes greater.

### 11 Circuits

In a simple circuit with a coil with self inductance L and negligible resistance and a capacitor with capacitance C, the charge  $\psi$  on the capacitor obeys

$$L\ddot{\psi} = -\frac{1}{C}\,\psi\tag{75}$$

That is  $\ddot{\psi} = -\omega_0^2 \psi$  with

$$\omega_0 = \sqrt{\frac{1}{LC}}.\tag{76}$$

This is the basis of lots of useful circuits, as, for example, in a radio. For more on the LC circuit, see pages 102-103 of French.



Problem 11.1 The figure shows an LC circuit with a battery to get it started. The capacitor is first charged to a voltage  $V_1$  by means of the battery. At time t = 0, the switch is thrown to connect the charged capacitor across the coil. derive the amplitude and phase constant of the resulting oscillation in the charge on the capacitor.

## 12 Why we might see oscillators often

Suppose that a mass m can move in one dimension, labelled by a coordinate r. Then the equation of motion is

$$m\ddot{r} = -\frac{d}{dr}V(r) \tag{77}$$

where V(r) is the potential energy. Let V(r) have a minimum at r = R. (That's a pretty common situation.) Define  $\psi = r - R$ . Then for small  $\psi$  we can write

$$V(R + \psi) \sim V(R) + \left[\frac{dV(r)}{dr}\right]_{r=R} \psi + \frac{1}{2} \left[\frac{d^2 V(r)}{dr^2}\right]_{r=R} \psi^2 + \frac{1}{3!} \left[\frac{d^3 V(r)}{dr^3}\right]_{r=R} \psi^3 + \cdots$$
(78)

(This is the Taylor expansion.) The condition that V(r) has a minimum at r = R implies that

$$\left[\frac{dV(r)}{dr}\right]_{r=R} = 0 \tag{79}$$

and also that

$$\left[\frac{d^2 V(r)}{dr^2}\right]_{r=R} > 0 \tag{80}$$

(or possibly that the second derivative equals zero, but we assume that that doesn't happen).

Now suppose that the particle is pushed just a little bit from its equilibrium position. What will happen? Well, it can't go very far because it hasn't got enough energy. Thus  $\psi$  will be non-zero, but it will be small. We are thus entitled to make the approximation of neglecting the  $\psi^3$ ,  $\psi^4$ , etc terms in the Taylor expansion of the potential. This gives

$$V(R+\psi) \approx V(R) + \frac{1}{2}s\psi^2,$$
(81)

where

$$s = \left[\frac{d^2 V(r)}{dr^2}\right]_{r=R}.$$
(82)

With this approximation, the force is

$$-\frac{d}{d\psi}V(R+\psi) = -s\psi.$$
(83)

This gives the equation of motion

$$m\ddot{\psi} = -s\psi,\tag{84}$$

so we have a harmonic oscillator. This is an approximation, and the approximation is that the particle is near its equilibrium position. If we supply lots of energy to the particle instead of just a little, the approximation will break down.

The pendulum is an example of this. The equation of motion is

$$ml^2\ddot{\theta} = -\frac{d}{d\theta}V(\theta) \tag{85}$$

where

$$V(\theta) = mgl[1 - \cos(\theta)].$$
(86)

Here I have used an angle instead of a position as the coordinate, so you may not have seen the equation of motion written like this. But it is easy to check that this equation amounts to

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta). \tag{87}$$

which is what we had previously in Eq. (??). We have

$$V(\theta) = mgl\left[\frac{1}{2}\theta^2 - \frac{1}{4!}\theta^4 + \cdots\right]$$
(88)

If we keep just the first term, we have the equation for simple harmonic motion. That will be an approximation to the exact equation. The approximation should be good for a pendulum that is not disturbed much from equilibrium.

Problem 12.1 Use a numerical solution of the exact equation and the approximate equation for a pendulum to examine the motion for a pendulum that is started from rest with an initial angle  $\theta_0 = \pi/2$ . (Just set m = g = l = 1.) Is the period of the pendulum longer or shorter than  $T_0 \equiv 2\pi/\omega_0$ ? By how much? That is, what is the approximate numerical value of  $T/T_0$ ? (Hint: I recommend that you make a graphs of the motion for the pendulum and for the analogous simple harmonic oscillator and read approximate values of Tand  $T_0$  off of your graph. Mathematica knows about  $\pi$ , which can be entered as Pi.)

## 13 Molecular vibrations

As an example, consider an ionic molecule made of two atoms separated by a distance r. Just to keep things simple, we suppose that one of the atoms is much lighter than the other, so that the heavy atom remains fixed and the light one can move around. For example, we may consider H Cl. We further simplify the problem by supposing that the light atom can move in only one dimension.

Let the light atom have mass m and a potential energy

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{B}{r^9}$$
(89)

The first term gives an attractive Coulomb force. The second gives a repulsion at short distances. There is an equilibrium at r = R given by

$$0 = +\frac{e^2}{4\pi\epsilon_0 R^2} - \frac{9B}{R^{10}} \tag{90}$$

Thus

$$B = \frac{e^2 R^8}{36\pi\epsilon_0} \tag{91}$$

The stiffness coefficient is the second derivative of V(r) at the minimum,

$$s = -\frac{2e^2}{4\pi\epsilon_0 R^3} + \frac{90B}{R^{11}} = \frac{2e^2}{\pi\epsilon_0 R^3}.$$
(92)

Thus if we let  $r = R + \psi$ , the equation of motion for  $\psi$  is

$$m\ddot{\psi} = -s\psi,\tag{93}$$

which gives vibrations with angular frequency

$$\omega_0 = \sqrt{\frac{s}{m}} = \sqrt{\frac{2e^2}{m\pi\epsilon_0 R^3}}.$$
(94)

Chemists study molecular vibrations with lasers: our model atom will absorb laser light with angular frequency  $\omega_0$ .

## 14 Damping

We have studied the harmonic oscillator with mass m and a restoring force  $-s\psi$ ,

$$m\ddot{\psi} = -s\psi. \tag{95}$$

Let us now add a frictional force  $-b\dot{\psi}$ :

$$m\ddot{\psi} = -s\psi - b\dot{\psi}.\tag{96}$$

This is a model that applies approximately to lots of physical systems. The idea is that "friction" acts to slow things down when they are moving. If the frictional force is a function of  $F(\dot{\psi})$  then expanding it in powers of  $\dot{\psi}$  we should have  $F(\dot{\psi}) = c_0 + c_1 \dot{\psi} + c_2 \dot{\psi}^2 + \cdots$ . We want  $c_0 = 0$  so that F(0) = 0. (Whatever force acts when  $\dot{\psi} = 0$  is included in the "spring force.") Then we want  $c_1 < 0$  so that the friction force slows things down rather than the reverse. (Frictional forces remove macroscopic potential and kinetic energy from the system and turn it into microscopic vibrations or "heat energy." If  $c_1$  were positive, the body in question would gain kinetic energy from the microscopic heat energy, in violation of the second law of thermodynamics.)

Finally, if  $\dot{\psi}$  is small, we can neglect the  $\dot{\psi}^2$  and higher order terms. This is not such a great approximation for a block sliding on a table, but it should be good if we spread a layer of oil between the block and the table because forces of viscosity in fluids are proportional to velocity differences.

We can divide by m in Eq. (??) to obtain

$$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = 0, \tag{97}$$

where  $\omega_0 = \sqrt{s/m}$  and  $\gamma = b/m$ .

What are the solutions of this? To find out, try the ansatz

$$\psi = C \exp(pt). \tag{98}$$

(If you have taken a course in differential equations, using this ansatz is what you learned to do for a linear differential equation with constant coefficients.) There are three points to address. First, this may seem too restrictive. In the end, in order to match initial conditions for  $\psi(0)$  and  $\dot{\psi}(0)$  we will need a more complicated solution. However, we have a *linear* equation, so that if  $\psi_1(t)$  is a solution and  $\psi_2(t)$  is a solution, then  $\psi_1(t) + \psi_2(t)$  is also a solution. Thus we have a chance to build up a complicated solution from simple solutions. Second, we will allow ourselves the option that C and pmight be complex. We can get a real solution by adding complex solutions. Third, we need to know how to differentiate  $\exp(pt)$ . That's easy. As we discussed in the section about complex numbers,

$$\frac{d}{dt}\exp(pt) = p\exp(pt).$$
(99)

This works even if p is complex.

We are now ready to use our ansatz in the differential equation. The differential equation holds if

$$p^{2}C\exp(pt) + \gamma pC\exp(pt) + \omega_{0}^{2}C\exp(pt) = 0.$$
 (100)

Equivalently, the differential equation holds if

$$p^2 + \gamma p + \omega_0^2 = 0. \tag{101}$$

Our differential equation has become an algebraic equation. Furthermore, we know the solutions:

$$p = -\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}.$$
 (102)

There are two cases to consider (plus a case on the dividing line).

Case I:  $\gamma^2 > 4\omega_0^2$ . We can call this heavy damping. There are two possible values for p,

$$-p_1 \equiv \mu_1 = \frac{1}{2}\gamma + \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}, -p_2 \equiv \mu_2 = \frac{1}{2}\gamma - \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}.$$
 (103)

We have two simple solutions,

$$C_1 \exp(-\mu_1 t)$$

$$C_2 \exp(-\mu_2 t) \tag{104}$$

We can add these two simple solutions to obtain

$$\psi(t) = C_1 \exp(-\mu_1 t) + C_2 \exp(-\mu_2 t) \tag{105}$$

We take  $C_1$  and  $C_2$  to be real so that  $\psi(t)$  is real. We can adjust the values of  $C_1$  and  $C_2$  to fit the initial conditions. The solution displays exponential decay; in fact, two kinds of exponential decay with different decay constants.

Case I:  $\gamma^2 < 4\omega_0^2$ . We can call this light damping. There are two possible values for p. If we define

$$\omega_f = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2} \tag{106}$$

then the two solutions are

$$p_1 = -\frac{1}{2}\gamma + i\omega_f,$$
  

$$p_2 = -\frac{1}{2}\gamma - i\omega_f.$$
(107)

We have two simple solutions,

$$C_1 \exp(-\frac{1}{2}\gamma t) \exp(i\omega_f t)$$

$$C_2 \exp(-\frac{1}{2}\gamma t) \exp(-i\omega_f t)$$
(108)

We can add these two simple solutions to obtain

$$\psi(t) = \exp(-\frac{1}{2}\gamma t) \left[C_1 \exp(i\omega_f t) + C_2 \exp(-i\omega_f t)\right]$$
(109)

We can make  $\psi(t)$  real by choosing  $C_2$  to be the complex conjugate of  $C_1$ , which we call simply C. Then our solution is

$$\psi(t) = \exp(-\frac{1}{2}\gamma t) \left[C \exp(i\omega_f t) + C^* \exp(-i\omega_f t)\right].$$
(110)

We can write this without any complex numbers if we take  $C = \frac{1}{2}A \exp(i\phi)$ ,  $C^* = \frac{1}{2}A \exp(-i\phi)$ , where A and  $\phi$  are real. then

$$\psi(t) = A \exp(-\frac{1}{2}\gamma t) \cos(\omega_f t + \phi). \tag{111}$$

Thus we have oscillations but modified by an exponential decay factor. The oscillations slowly die away.

Notice how useful complex numbers have been for this analysis.

Problem 14.1 Consider an oscillator with mass m = 1.0 kg attached to a spring with stiffness s = 64 N/m. The mass slides on some oil, so that the frictional force has the form  $-b d\psi/dt$ . (a) What value of b would make the amplitude decrease from A to A/e in a time 10 s? (b) One often defines the Q-value of on oscillator as  $\omega_0/\gamma$ . That way if the oscillator is very lightly damped, its Q is big. What is the Q value for the oscillator with this value of b? (c) What value for b would put the oscillator just at the boundary between light damping and heavy damping? (The boundary is called critical damping.)

Problem 14.2 Show that the amplitude of a damped vibration is halved in a time  $1.39/\gamma$ .

Problem 14.3 Show that the successive maxima of  $\psi$  for a damped oscillator are separated in time by  $\Delta t = 2\pi/\omega_f$ .

### 15 Oscillator circuit with damping

We have studied the electric LC oscillator circuit, in which there is a coil with self-inductance L in series with a capacitor with capacitance C. Now lets add a resistor with resistance R in series with these. The potential drop across the coil is LdI/dt, the potential drop across the resistor is RI, and the potential drop across the capacitor is Q/C. The total potential drop around the circuit is zero, so

$$L\frac{dI}{dt} + RI + Q/C = 0.$$
(112)

We also recognize that I = dQ/dt, so that this equation is

$$L\frac{d^{2}Q}{dt^{2}} + R\frac{dQ}{dt} + Q/C = 0.$$
 (113)

To make this look like our standard oscillator equation, we denote Q by  $\psi$ . Then

$$L\ddot{\psi} + R\dot{\psi} + (1/C)\,\psi = 0.$$
(114)

We divide by L and denote

$$\omega_0 = \sqrt{\frac{1}{LC}} 
\gamma = \frac{R}{L}$$
(115)

Then

$$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = 0. \tag{116}$$

We thus have the standard differential equation for a damped oscillator. For light damping,  $\gamma^2 < 4\omega_0^2$ , the solution is

$$\psi(t) = A \exp(-\frac{1}{2}\gamma t) \cos(\omega_f t + \phi), \qquad (117)$$

where

$$\omega_f = \sqrt{\omega_0^2 - \gamma^2/4} \tag{118}$$

The circuit oscillates, but the oscillations die out with a characteristic time  $2/\gamma$ .



Problem 15.1 The figure shows an LRC circuit with a battery to get it started. The capacitor is first charged to a voltage  $V_1$  by means of the battery. At time t = 0, the switch is thrown to connect the charged capacitor across the coil. Find the charge Q(t) on the capacitor as a function of time t for

t > 0. What is the condition on R compared for a given L and C such that the circuit oscillates instead of having Q just go to zero without oscillations? For R such that the current does oscillate, sketch the qualitative behavior of Q(t) exhibited by your solution.

### 16 Forced vibrations

What if we take a damped oscillator and subject it to a force that varies with time proportionally to  $\cos(\omega t)$ ? Here we have in mind that  $\omega$  is under our control. It might be close to the natural frequency of the oscillator, or it might not.

For a mechanical oscillator, let the force be  $F_0 \cos(\omega t)$ , so that the differential equation is

$$m\psi + b\psi + s\psi = F_0 \cos(\omega t) \tag{119}$$

Dividing through by m, defining  $f_0 = (F_0/m)$ , and using our previous definitions for  $\omega_0$  and  $\gamma$ , this is

$$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = f_0 \cos(\omega t). \tag{120}$$

This same equation arises in other circumstances also.

How can we solve Eq. (??)? To start, we will look for a "steady state" solution. Specifically, we will look for a solution of the form

$$\psi(t) = A\cos(\omega t + \phi). \tag{121}$$

We call this "steady state" because the oscillator oscillates forever with the angular frequency of the driving force. Later, we will see that there are solutions in which the oscillator does something else for awhile before settling down to the steady state solution.

OK, how do we find a solution of the form (??)? We have to show that for a certain amplitude A and phase  $\phi$ , this  $\psi(t)$  actually solves the differential equation (??). In addition, we have to find A and  $\phi$ . There is an easy way to do this. Look for a solution  $\eta(t)$  of the equation

$$\ddot{\eta} + \gamma \dot{\eta} + \omega_0^2 \eta = f_0 \exp(i\omega t) \tag{122}$$

of the form

$$\eta(t) = C \exp(i\omega t). \tag{123}$$

Here C is a complex constant. Once we have found  $\eta(t)$ , break it up into its real and imaginary parts:

$$\eta(t) = \psi(t) + i\lambda(t). \tag{124}$$

Then we will have

$$\ddot{\psi} + i\ddot{\lambda} + \gamma\dot{\psi} + i\gamma\dot{\lambda} + \omega_0^2\psi + i\omega_0^2\lambda = f_0\cos(\omega t) + if_0\sin(\omega t).$$
(125)

Since two complex numbers are equal if both their real and their imaginary parts are equal, this is actually two equations,

$$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = f_0 \cos(\omega t), \ddot{\lambda} + \gamma \dot{\lambda} + \omega_0^2 \lambda = f_0 \sin(\omega t).$$
(126)

The first equation tells us that  $\psi(t)$  is the function we want. We can just throw  $\lambda(t)$  away.

Recall that the form of  $\eta$  that we are looking for is

$$\eta(t) = C \exp(i\omega t). \tag{127}$$

Let us write the constant C as

$$C = A \exp(i\phi) \tag{128}$$

with A and  $\phi$  real. Then

$$\eta(t) = A \exp(i[\omega t + \phi]) = A \cos(\omega t + \phi) + iA \sin(\omega t + \phi).$$
(129)

Thus

$$\psi = A\cos(\omega t + \phi), \tag{130}$$

which is the form that we were seeking.

We thus see that we should solve

$$\ddot{\eta} + \gamma \dot{\eta} + \omega_0^2 \eta = f_0 \exp(i\omega t) \tag{131}$$

with

$$\eta(t) = C \exp(i\omega t). \tag{132}$$

That's really easy. We get

$$-\omega^2 C \exp(i\omega t) + i\omega\gamma C \exp(i\omega t) + \omega_0^2 C \exp(i\omega t) = f_0 \exp(i\omega t).$$
(133)

That is

$$\left[-\omega^2 + i\omega\gamma + \omega_0^2\right]C = f_0. \tag{134}$$

The differential equation is solved if the complex constant C is

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma}.$$
(135)

We can see immediately that C is big if we choose  $\omega$  so that the denominator is small,  $\omega \approx \omega_0$ . This is the most important lesson about the driven oscillator.

To see what is happening in more detail, we should write C in the form (??). To do that multiply and divide the expression for C by the complex conjugate of the denominator:

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + i\omega\gamma} \frac{\omega_0^2 - \omega^2 - i\omega\gamma}{\omega_0^2 - \omega^2 - i\omega\gamma} = \frac{f_0 \left[\omega_0^2 - \omega^2 - i\omega\gamma\right]}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}.$$
 (136)

Thus

$$C = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \left\{ \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} + i \frac{-\omega\gamma}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}} \right\}.$$
(137)

This has the form

$$C = A \left\{ \cos(\phi) + i \sin \phi \right\}, \tag{138}$$

where

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$$
(139)

and

$$\cos(\phi) = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}, \quad \sin(\phi) = \frac{-\omega\gamma}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}.$$
 (140)

We say that the oscillator is "in resonance" when  $\omega = \omega_0$ . Then  $\phi = -\pi/2$  and A is almost maximum. If  $\gamma$  is small, then the amplitude is very large near resonance.

Problem 16.1 A system with m = 0.1 kg, s = 1.6 N/m and b = 0.1 kg/s is driven by a harmonically varying force with amplitude 2 N. Find the

amplitude and phase of the steady state motion when the angular frequency of the driving force is (a)  $\omega = 0.4 \text{ s}^{-1}$ , (b)  $\omega = 4 \text{ s}^{-1}$ , (c)  $\omega = 40 \text{ s}^{-1}$ .

Problem 16.2 If  $\gamma$  is fairly small, the amplitude A in Eq. (??) is big when  $\omega \approx \omega_0$ . For which value of  $\omega$  is it biggest? What is the value of A at this value of  $\omega$ ?

### 17 Energy in the driven oscillator

Let's review the driven oscillator. Suppose that we have a mechanical oscillator with a body of mass m attached to a spring of stiffness s, with a frictional force -b times the velocity. Now we drive the system with a force  $F_0 \cos \omega t$ . Then the displacement  $\psi$  of the body obeys

$$m\ddot{\psi} + b\dot{\psi} + s\psi = F_0\cos(\omega t). \tag{141}$$

We thus have

$$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = f_0 \cos(\omega t). \tag{142}$$

with  $\gamma = b/m$ ,  $\omega_0^2 = s/m$ , and  $f_0 = F_0/m$ . We know the steady state solution of this,

$$\psi(t) = A\cos(\omega t + \phi). \tag{143}$$

Here the amplitude A is

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}.$$
 (144)

Also,

$$A\sin(\phi) = -\frac{\omega\gamma f_0}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}.$$
(145)

We will use this later. We see that A is much big if  $\gamma/\omega_0$  is small and  $\omega$  is close to  $\omega_0$ .

Now we can look at the energy stored in the oscillator. The energy is

$$W = \frac{1}{2}m\dot{\psi}^2 + \frac{1}{2}s\psi^2.$$
 (146)

That is,

$$W = \frac{mA^2}{2} \left\{ \omega^2 \sin^2(\omega t + \phi) + \omega_0^2 \cos^2(\omega t + \phi) \right\}.$$
 (147)

For  $\omega = \omega_0$ , W is a constant [since  $\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) = 1$ ], but in general, it is a constant part plus a part that oscillates. We can easily find the average value,  $\langle W \rangle$ , of W, by averaging over one period. We use

$$\langle \sin^2(\omega t + \phi) \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \sin^2(\theta) = \frac{1}{2}.$$
 (148)

and

$$\left\langle \cos^2(\omega t + \phi) \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, \cos^2(\theta) = \frac{1}{2}.$$
 (149)

This gives

$$\langle W \rangle = \frac{mA^2}{4} \left\{ \omega^2 + \omega_0^2 \right\}. \tag{150}$$

We can substitute from Eq. (??) for A to get

$$\langle W \rangle = \frac{mf_0^2}{4} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}.$$
 (151)

This is perhaps a little complicated, but it does show us that if  $\gamma$  is small, the stored energy is biggest when  $\omega$  is near to  $\omega_0$ . At  $\omega = \omega_0$ ,  $\langle W \rangle$  is

$$\langle W \rangle = \frac{m f_0^2}{2\omega_0^2} Q^2, \tag{152}$$

where  $Q = \omega_0 / \gamma$ . That's big if Q is big.

Since there is friction in the oscillator, it is losing energy. But it is gaining the same amount of energy from the driving force. We can work out the rate at which the driving force delivers energy. It is

$$P = F_0 \cos(\omega t) \,\psi. \tag{153}$$

That is

$$P = -mf_0 \cos(\omega t) A\omega \sin(\omega t + \phi)$$
  
=  $-mf_0 A\omega \cos(\omega t) [\cos(\omega t) \sin(\phi) + \sin(\omega t) \cos(\phi)]$  (154)

This oscillates about its average value. It is easy to work out what its average value is if we use

$$\langle \cos(\omega t)\sin(\omega t)\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \ \cos(\theta) \sin(\theta) = 0$$
 (155)

in addition to our previous relations. We get

$$\langle P \rangle = -\frac{1}{2} m f_0 A \omega \sin(\phi) = \frac{m f_0^2}{2} \frac{\omega^2 \gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$
 (156)

where we have used Eq. (??) in the second step. The amount of energy that we add per cycle of the oscillator is

$$\Delta W \equiv \langle P \rangle \frac{2\pi}{\omega} = \frac{mf_0^2}{2} \frac{2\pi\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}.$$
 (157)

If  $\gamma$  is small, we see that we have to put in a lot of energy per cycle if  $\omega$  is close to  $\omega_0$ . If we choose  $\omega = \omega_0$ , then

$$\Delta W = \frac{mf_0^2}{2\omega_0^2} 2\pi Q.$$
 (158)

The ratio of the total energy to the energy added (and lost) per cycle to the total energy is

$$\frac{\Delta W}{\langle W \rangle} = \frac{2\pi}{Q}.$$
(159)

Compare this to what will happen if we now stop the driving force. We will have oscillations of the form

$$\psi(t) = A\cos(\omega_f t + \phi) e^{-\gamma t/2}.$$
(160)

Then

$$\dot{\psi}(t) = -\omega_f A \sin(\omega_f t + \phi) e^{-\gamma t/2} - \frac{1}{2} \gamma A \cos(\omega_f t + \phi) e^{-\gamma t/2}.$$
 (161)

We can calculate the energy as a function of time. It will have some little wiggles in it, but each term in the energy has a factor

$$e^{-\gamma t}$$
. (162)

[Note that it is  $\gamma t$  and not  $\gamma t/2$  in the exponent because we have to square  $\psi$  and  $\dot{\psi}$  to get terms in the energy.] Thus in one cycle,  $t = 2\pi/\omega_0$ , the oscillator loses a fraction

$$\frac{2\pi\gamma}{\omega_0} = \frac{2\pi}{Q} \tag{163}$$

of its energy. When we are driving the oscillator, its amplitude doesn't decay because we keep putting just this amount of energy back in.

Problem 17.1 Consider the average energy  $\langle W \rangle$  in a driven oscillator. Show that this energy is mostly potential energy when  $\omega \ll \omega_0$  and is mostly kinetic energy when  $\omega \gg \omega_0$ , while it is an equal mixture of potential energy and kinetic energy when  $\omega = \omega_0$ .

# 18 The driven LRC circuit

An important example of everything that we have learned is the driven LRC circuit. The study of this example can serve as a review.

Imagine three circuit elements attached in series: a coil with inductance L, a resistor with resistance R, and a capacitor with capacitance C. Now attach this string of circuit elements to a voltage source that supplies an oscillating potential difference  $V = V_0 \cos(\omega t)$  between its terminals, independent of how much current is flowing through it. (We say that the source has no internal resistance.) We then have

$$L\ddot{\psi} + R\dot{\psi} + \frac{1}{C}\psi = V_0\cos(\omega t) \tag{164}$$

where  $\psi$  is the charge on the capacitor and  $\dot{\psi}$  is the current in the circuit. Dividing by L, we have

$$\ddot{\psi} + \gamma \dot{\psi} + \omega_0^2 \psi = f_0 \cos(\omega t) \tag{165}$$

where

$$\gamma = \frac{R}{L}$$

$$\omega_0^2 = \frac{1}{LC}$$

$$f_0 = \frac{V_0}{L}.$$
(166)

We know from our previous analysis what this circuit will do. First, there is a steady state solution,

$$\psi_s(t) = A\cos(\omega t + \phi) \tag{167}$$

with A given by Eq. (??) and  $A\sin(\phi)$  given by Eq. (??) [except that now the parameters  $\omega_0$ ,  $\gamma$  and  $f_0$  refer to circuit elements instead of springs].

We also know that if f = 0 the general solution of Eq. (??) is

$$\psi_f(t) = A_f \cos(\omega_f t + \phi_f) e^{-\gamma t/2}$$
(168)

in the case that  $\gamma < 4\omega_0^2$ . Here

$$\omega_f = \sqrt{\omega_0^2 - \gamma^2/4}.$$
(169)

The two real parameters  $A_f$  and  $\phi_f$  can be adjusted to fit the initial conditions. In the alternative that  $\gamma > 4\omega_0^2$ , the general solution of Eq. (??) is

$$\psi_f(t) = C_1 e^{-\mu_1 t} + C_2 e^{-\mu_1 t} \tag{170}$$

where

$$\mu_{1} = \frac{1}{2}\gamma + \sqrt{\frac{1}{4}\gamma^{2} - \omega_{0}^{2}}, \mu_{1} = \frac{1}{2}\gamma - \sqrt{\frac{1}{4}\gamma^{2} - \omega_{0}^{2}}.$$
(171)

Now in the case that we have before us,  $f_0$  is not zero. Nevertheless, the free solutions are still useful. Think of our equation as

$$\mathcal{D}\psi = f \tag{172}$$

where

$$\mathcal{D} = \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \tag{173}$$

and f stands for the function

$$f(t) = f_0 \cos(\omega t). \tag{174}$$

Then we have one function  $\psi_s$  that satisfies  $\mathcal{D}\psi_s = f$  and we have another function  $\psi_f$ , which contains two adjustable parameters, that satisfies  $\mathcal{D}\psi_f = 0$ . Thus if we put

$$\psi(t) = \psi_s(t) + \psi_f(t) \tag{175}$$

we will have

$$\mathcal{D}\psi = \mathcal{D}(\psi_s + \psi_f) = \mathcal{D}\psi_s + \mathcal{D}\psi_f = f + 0 = f.$$
(176)

Thus Eq. (??) gives us a solution of Eq. (??). Furthermore, this solution has two adjustable parameters [either  $A_f$  and  $\phi_f$  or  $C_1$  and  $C_2$ ]. We can use these parameters to fit whatever initial conditions might apply.

Note, finally, that the free oscillation part of the solution,  $\psi_f(t)$ , fades away after awhile because of its factor  $\exp(-\gamma t/2)$ .

This pretty much summarizes everything except how to solve  $\mathcal{D}\psi = f$ and  $\mathcal{D}\psi = 0$ . Personally, I don't remember the solutions. Its easier to simply remember how to get them.

To solve  $\mathcal{D}\psi = 0$ , just assume a solution of the form

$$\psi = Ce^{pt}.\tag{177}$$

Then you get an algebraic equation

$$p^2 + \gamma p + \omega_0^2 = 0. \tag{178}$$

Solving this gives one or the other of the two solutions depending on the sign of  $\gamma^2 - 4\omega_0^2$ . In the case  $\gamma^2 < 4\omega_0^2$ , the possible values of p are complex, so the simple solution  $Ce^{pt}$  is complex. A real valued solution can be obtained by adding two complex solutions. Another strategy, which amounts to the same thing, is to simply take the real part of the complex solution.

To solve  $\mathcal{D}\psi = f$ , we first replace f by  $f_0 \exp(i\omega t)$ . At the end, we will take the real part of our solution to get the solution to the original equation. Then we assume a steady state solution of the form  $C \exp(i\omega t)$  with a complex constant C. This gives an algebraic equation

$$[-\omega^2 + i\gamma\omega + \omega_0^2]C = f_0.$$
(179)

We just solve this for C.

Problem 18.1 Find the solution of Eq. (??) for the initial conditions  $\psi(0) = 0$ ,  $\dot{\psi}(0) = 5 \text{ C s}^{-1}$ . Take  $\omega = \omega_0 = 100 \text{ s}^{-1}$ ,  $\gamma = 1 \text{ s}^{-1}$ ,  $f_0 = 3 \text{ C s}^{-2}$ . ("C" here is Coulombs.) Note that the steady state solution is particularly simple for  $\omega = \omega_0$ . If you use this simplicity, you won't get too bogged down with algebra.