Physics 351 Foundations of Physics II Davison E. Soper

December 7, 2003

1 Electromagnetic waves

To really learn this, you should take a course in electricity and magnetism. But here is a simplified version. This should suffice for understanding simple optics.

There is an *electric field* \vec{E} and there is a *magnetic induction* \vec{B} . There are two constants that characterize a material through which an electromagnetic wave might be moving, μ and ϵ . In vacuum, these have values μ_0 and ϵ_0 respectively. In most materials of interest in optics, μ is very close to μ_0 , but ϵ can differ quite a lot from ϵ_0 . For our purposes, it is convenient to state the equations in terms of the *magnetic field* $\vec{H} \equiv \vec{B}/\mu$. (Actually, \vec{B} is the more basic quantity and deserves to be called the magnetic field. The names are a historical accident. People often refer to the magnetic field \vec{B} .)

We consider an electromagnetic plane wave moving in the z-direction. Then the electric field points in some direction transverse to the z-direction – let's say the $\pm x$ direction. The magnetic field points in a direction perpendicular to both the propagation direction and the direction of \vec{E} . In our case, \vec{B} points in the $\pm y$ direction. The electric and magnetic fields are (in general) related by

$$\frac{\partial E_1}{\partial x_3} - \frac{\partial E_3}{\partial x_1} = -\mu \frac{\partial H_2}{\partial t}.$$
(1)

There are two more equations that come along with this one. They are obtained by taking cyclic permutations of the indices 1,2,3. For the case of a plane wave in the 3 direction, this reduces to

$$\frac{\partial E_1}{\partial x_3} = -\mu \frac{\partial H_2}{\partial t}.$$
(2)

The electric and magnetic fields are also (in general) related by

$$\frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} = +\epsilon \frac{\partial E_1}{\partial t}.$$
(3)

There are two more equations that come along with this one, obtained by taking cyclic permutations of the indices 1,2,3. For the case of a plane wave in the 3 direction, this reduces to

$$\frac{\partial H_2}{\partial x_3} = -\epsilon \frac{\partial E_1}{\partial t}.$$
(4)

Let's see what the implications are. Differentiate Eq. (2) with respect to x_3 :

$$\frac{\partial^2 E_1}{\partial x_3^2} = -\mu \frac{\partial^2 H_2}{\partial t \partial x_3}.$$
(5)

Differentiate Eq. (4) with respect to t:

$$\frac{\partial^2 H_2}{\partial x_3 \partial t} = -\epsilon \frac{\partial^2 E_1}{\partial t^2}.$$
(6)

Comparing these gives

$$\frac{\partial^2 E_1}{\partial x_3^2} = \epsilon \mu \frac{\partial^2 E_1}{\partial t^2} \tag{7}$$

or

$$\frac{\partial^2 E_1}{\partial t^2} - c^2 \frac{\partial^2 E_1}{\partial x_3^2} = 0, \tag{8}$$

where

$$c = \frac{1}{\sqrt{\epsilon\mu}}.$$
(9)

Thus we get waves. For vacuum, we get $c_0 = 1/\sqrt{\epsilon_0 \mu_0} \approx 3 \times 10^8 \text{m/s.}$ (This speed is usually called just *c*.)

James Clerk Maxwell invented the equations that we used as input (plus two more), based on experimental information that had been discovered by that time. Surely he was astonished when he found this result: waves with velocity c_0 .

The ratio c_0/c for a material is called the *index of refraction n*:

$$c = \frac{c_0}{n} \tag{10}$$

For instance, for water, n is about 1.3. For air, $n \approx 1$.

These are transverse waves: \vec{E} can point in any direction transverse to the propagation direction. We say that the wave can be polarized in any transverse direction.

Electromagnetic waves can be reflected and transmitted at an interface between two materials. We can obtain the rules for reflection and transmission by working with analogy to the other waves we have studied. To be definite, let's choose sound waves. For a sound wave coming to an interface (with normal incidence) we match the displacement ψ and the excess force per unit area f. ($f = T_{33} - T_{33}^0$ is the change in the pressure above the ambient pressure. One would normally call it p or ΔP , but f fits with our standard notation.) These two quantities are related by

$$f = -Zc\frac{\partial\psi}{\partial z}.$$
(11)

This relation, which defined Z for us, can be rewritten as

$$\frac{\partial f}{\partial t} = -Zc \frac{\partial}{\partial z} \left(\frac{\partial \psi}{\partial t} \right). \tag{12}$$

What do we match for electromagnetic waves? For waves moving perpendicular to the interface, the rule is that E and H are the same on one side of the interface as on the other. We may consider that matching H is like matching $\partial \psi / \partial t$ and matching E is like matching $T_{33} - T_{33}^0 = f$ in a sound wave. This correspondence,

$$\begin{array}{lcl} H & \rightarrow & \frac{\partial \psi}{\partial t} \\ E & \rightarrow & f \end{array}$$
 (13)

seems a little contrived, but it makes everything correspond. We have

$$\frac{\partial E_1}{\partial t} = -\sqrt{\frac{\mu}{\epsilon}} \ c \frac{\partial H_2}{\partial z}.$$
(14)

Compare this to Eq. (12), we see that the impedance for electromagnetic waves is

$$Z = \sqrt{\frac{\mu}{\epsilon}}.$$
(15)

Most materials have $\mu \approx \mu_0$. In that case

$$Z \approx \frac{Z_0}{n} \tag{16}$$

where *n* is the index of refraction and $Z_0 = \sqrt{\mu_0/\epsilon_0}$.

This analogy works for power transmitted too. The power per unit area transmitted in a sound wave is

$$\frac{P}{\mathcal{A}} = f \,\frac{\partial \psi}{\partial t}.\tag{17}$$

The analogue for electromagnetic waves is

$$\frac{P}{\mathcal{A}} = E H. \tag{18}$$

A little more precisely, the power transmitted per unit area in an electromagnetic wave has the name "Poynting's vector" \vec{S} . The power transmitted per unit area in the 3-direction is

$$S_3 = E_1 H_2 - E_2 H_1. (19)$$

This is by no means obvious. We would have to delve quite deeply into electrodynamics to derive it. Nevertheless, it is nice that the analogy works so nicely.

Problem 1.1 An light wave in air strikes the surface of some water. The light wave is directed perpendicular to the water surface. Some of the light enters the water, some is reflected. What is the reflection coefficient R (the ratio of the electric field amplitude in the incident wave to the electric field amplitude in the reflected wave)? What fraction of the incident power in the incoming light wave is reflected? If the light wave moves from water to air, what fraction of the incident power is reflected?

2 Reflection and transmission in three dimensions

Suppose that we have material A in the region z < 0 and material B in the region z > 0. An incident wave

$$\vec{E}_i(t,\vec{x}) = \vec{E}_{0,i} \cos(\omega t - \vec{k}_i \cdot \vec{x})$$
(20)

comes from the region z < 0 and hits the surface. What happens?

First of all, note that this is a sinusoidal wave with wave fronts perpendicular to the unit vector \hat{k}_i in the direction of \vec{k}_i ,

$$\hat{k}_{i} = \frac{1}{k_{i}} \vec{k}_{i}$$

$$k_{i} = \sqrt{\vec{k}_{i}^{2}}.$$
(21)

The wave fronts move in the direction \hat{k}_i . The wave has the form

$$\vec{E}_i(t,\vec{x}) = \vec{E}_{0,i} \cos\left(\omega \left(t - \frac{1}{c_A}\hat{k}_i \cdot \vec{x}\right)\right)$$
(22)

where

$$c_A = \frac{\omega}{k_i} \tag{23}$$

Thus ω/k_i is the speed of the waves. This wave form will obey the wave equation provided c_A is the right wave speed for medium A,

$$c_A = c/n_A. \tag{24}$$

Here I use c for the speed of light in vacuum. According to our previous analysis of the Maxwell equations, $\vec{E}_{0,i}$ should be perpendicular to \hat{k}_i .

Well, what happens is that there is a reflected wave

$$\vec{E}_r(t,\vec{x}) = \vec{E}_{0,r} \cos(\omega t - \vec{k}_r \cdot \vec{x})$$
(25)

in the region z < 0. In order for this wave to satisfy the Maxwell equations, the magnitude k_r of $\vec{k_r}$ must be right,

$$k_r = n_A \frac{\omega}{c} = k_i. \tag{26}$$

There will also be a transmitted wave

$$\vec{E}_t(t,\vec{x}) = \vec{E}_{0,t} \cos(\omega t - \vec{k}_t \cdot \vec{x})$$
(27)

in the region z > 0. In order for this wave to satisfy the Maxwell equations, the magnitude k_r of $\vec{k_r}$ must be right,

$$k_t = n_B \frac{\omega}{c} = \frac{n_B}{n_A} k_i. \tag{28}$$

Thus the wavelength of the transmitted wave is different from the wavelength of the incident wave.

Note that the angular frequency ω is the same for all three waves. I have just built this in from the start, but we should not take it for granted. The reason that ω is the same for all three waves is that things have to match at the boundary. If 1027 wave fronts are incident at a point on the boundary, then it had better be the case that 1027 wave fronts are reflected and 1027 wave fronts are reflected.

Let's see if we can say something more. At $x_3 = 0$ the incident wave is

$$\vec{E}_i(t, \vec{x}) = \vec{E}_{0,i} \cos(\omega t - k_{i,1}x_1 - k_{i,2}x_2).$$
(29)

The reflected wave is

$$\vec{E}_r(t,\vec{x}) = \vec{E}_{0,r} \cos(\omega t - k_{r,1}x_1 - k_{r,2}x_2).$$
(30)

The transmitted wave is

$$\vec{E}_t(t,\vec{x}) = \vec{E}_{0,t} \cos(\omega t - k_{t,1}x_1 - k_{t,2}x_2).$$
(31)

There are matching conditions at z = 0 for \vec{E} and also for \vec{H} . The matching has to work for all t and for all x_1 and x_2 . I think that we have seen enough of boundary conditions already to appreciate without writing all of the equations out in detail that we need

$$k_{r,1} = k_{i,1}$$

$$k_{r,2} = k_{i,2}$$

$$k_{t,1} = k_{i,1}$$

$$k_{t,2} = k_{i,2}.$$
(32)

That is, the components of the wave vectors \vec{k} in the plane of the interface have to be the same for all three waves. Another way to say this is the following. The reason that k_1 and k_2 are the same for all three waves is that things have to match at the boundary. If at a given time there are in the incident wave 1027 wave fronts between two points on the boundary , then it had better be the case that there are in the reflected wave and also in the transmitted wave 1027 wave fronts between these two points on the boundary. We learn first of all that all of the vectors $\vec{k_i}$, $\vec{k_r}$ and $\vec{k_t}$ lie in the same plane, the plane formed by $\vec{k_i}$ and the normal to the surface. Let's let this be the 1-3 plane. Then our result is a little simpler

$$k_{r,1} = k_{i,1} k_{t,1} = k_{i,1}.$$
(33)

Then we can do some geometry: define the angles of incidence, reflection, and refraction by

$$k_{i,1} = k_i \sin \theta_i$$

$$k_{r,1} = k_r \sin \theta_r$$

$$k_{t,1} = k_t \sin \theta_t.$$
(34)

Then for the reflected wave we have

$$k_r \sin \theta_r = k_i \sin \theta_i$$

$$k_r = k_i.$$
(35)

This implies that

$$\sin \theta_r = \sin \theta_i. \tag{36}$$

The wave bounces off the surface making the same angle to the normal as the incident wave.

Then for the transmitted wave we have

$$k_t \sin \theta_t = k_i \sin \theta_i$$

$$k_t = \frac{n_B}{n_A} k_i.$$
(37)

This implies "Snell's Law,"

$$n_B \sin \theta_t = n_A \sin \theta_i. \tag{38}$$

The wave bounces off the surface making a different angle to the normal as the incident wave. For instance, in going from air to glass, which has $n_B > 1$, the wave vector is bent toward the vector normal to the surface. This has major implications for optics.

If a light wave goes from a medium with index of refraction n > 1 to a medium with an index of refraction that is nearly equal to 1, like air, an interesting phenomenon can occur. Suppose that

$$n\sin\theta_i > 1. \tag{39}$$

Then there can be no angle of transmission such that $\sin \theta_t = n \sin \theta_i$. Thus there is no transmitted wave. All of the incoming wave is reflected.

Actually, it's more interesting than that. You can solve a quadratic equation for the 3-component of \vec{k}_t . You get an imaginary number, $k_{t,3} = i\kappa$. Then instead of $\exp(ik_{t,3}x_3)$ you have $\exp(-\kappa x_3)$. This corresponds to a wave that penetrates a little way into the medium to the right of $x_3 = 0$, but dies off exponentially as you get away from the surface. However, we won't need to know this for our purposes in this course, so you should erase the contents of this paragraph from your mind.

Then phenomenon of the all of the energy being reflected when $n \sin \theta_i > 1$ is called *total internal reflection*. You can see it yourself by swimming under water while looking up. Then the surface of the water beyond a certain angle looks shiny, like a big mirror. Total internal reflection finds an important application in fiber optic fibers that carry light for quite long distances in a long, thin piece of plastic. It is significant that the fiber does not need to be perfectly straight. The light just bounces off the surface of the fiber as it travels from one end to the other.

3 Virtual images

With Snell's Law and the reflection law $\theta_r = \theta_i$ at hand, we are prepared to talk about images. There are two kinds of images, real images and virtual images. Here we discuss virtual images in the simplest case.

Consider the light coming from a point on some object. We assume that the size of everything we discuss is much greater than the wavelength of the light. We can think of light as travelling along rays coming from the point. The ray direction is perpendicular to the wave front direction. That is, the unit vector \hat{k} that we have used to describe infinite plane waves is the ray direction. Now our waves are not exactly infinite or plane, but on the scale of one wavelength they are almost infinite plane waves. When the light reaches a boundary between two different media, the ray is bent according to Snell's Law.

Suppose that light comes from a point in medium A with index of refraction n_A and enters medium B with index of refraction n_B . The rays in medium B are bent compared to the rays in medium A from which they came. If you trace the rays in medium B back as imaginary lines into medium A, they may (approximately) meet at a point. Then this point is called the "image" of the original point.



In this figure, I illustrate two sets of rays in medium B. One set of rays was originally emitted from the nose of a fish. The fish, however, is not shown. When these rays in medium B are extrapolated back into medium A, they come together at a point. This point is the virtual image of the nose of the fish. The other set of rays was originally emitted from the tail of the fish. When these rays in medium B are extrapolated back into medium A, they come together at a point. This point is the virtual image of the tail of the fish. Now consider lots of sets of rays like these. Each set of rays was originally emitted from part X of the fish and extrapolates back to a point that constitutes the virtual image of part X of the virtual fish. The virtual fish is shown in the figure. If you are looking at all of this from medium B, your eyes and brain put it all together and report the existence of a fish where the virtual image is. Your eyes and brain are wrong: there is only a

virtual fish there.

Problem 3.1 A physicist visits an aquarium in Osaka, Japan. He stands facing a wall made of plastic with index of refraction n = 1.33. Behind the wall is water with index of refraction n = 1.33. Fish are swimming in the water. The physicist sees a fish that *appears* to be 10 cm long swimming left to right, located 30 cm directly behind the front surface of the wall. Where is the fish really? How long is it really? Use small angle approximations.

It is easy to figure out where the virtual image is, or at least it is easy as long as the line from your eyes to the fish is approximately perpendicular to the air-plastic boundary surface. In this case, all angles are small and we can use $\tan \theta \approx \sin \theta \approx \theta$.

To find the image, just assume that the object (eg. the fish's eye) is a location (x, 0, z) with z < 0 (in the water). A light ray leaves the object and travels with angle θ until it comes to the boundary at z = 0 at x-coordinate x + h with $h \approx x + |z|\theta$. Then it is bent to a new angle $\theta' \approx n\theta$. The new ray can be traced back into the water, and we can look for the point (x, 0, z') where it intersects a ray that goes straight ahead, with zero angle, and is thus not deflected. From the figure, we see that $h \approx |z'|\theta'$. Thus $|z'| \approx h/\theta' \approx |z|\theta/\theta'$. Thus

$$z' \approx z\theta/\theta'$$
 geometry
 $\theta/\theta' \approx 1/n$ Snell's Law. (40)

We conclude that

$$z' = z/n. \tag{41}$$

We have used some approximations, so there are corrections when the angles are not small.

We also have, directly from the picture, that the x coordinate of the virtual image is the same as the x coordinate of the object,

$$x' = x. \tag{42}$$



4 Virtual images and real images

We have seen what a virtual image is. Suppose that light rays from a point object traverse an optical device. Consider the new light rays, that is the rays after they are bent by the optical device. Extend the new light rays backwards in straight lines. If they meet at a point, that point is the image of the original point object. We call this a virtual image.

Now suppose that the new light rays actually meet at a point. Then we call this point the image of the original object and we say that it is a real image.

A real or virtual image is something that you can see by putting your eye in the path of the rays. (In the case of a real image, your eye should be "downstream" from the real image.) Your eye & brain will see a point source of light where the image is. In the case that there is an extended object consisting of many point objects – a fish, say – there will be an extended image consisting of the many point images. They your eye/brain will see a fish where the extended image is.

In the case of a real image, but not a virtual image, you can also put a charge coupled device (CCD) where the image is and take a picture of it.

5 The thin lens

We consider a lens made of a material with index of refraction n. For the sake of definiteness, we consider that the surfaces of the lens are both convex. The left hand surface is a sphere with radius R_1 and the right hand surface is a sphere with radius R_2 . We suppose that the lens is thin enough that we can neglect its thickness with respect to other distances along the lens axis in the problem.

We want to see what kind of image our lens can make. For this purpose, we consider that the lens is located in the plane z = 0 in our coordinate system. We consider an object at (x, z) = (h, -d). (We take everything to be in the x, z plane, with y = 0 for this problem.) A light ray coming from this object follows a path

$$x = h + (z+d)\theta \tag{43}$$

where θ is the angle of the ray. At the lens, z = 0, x is

$$x_L = h + d\,\theta. \tag{44}$$

The lens bends the ray to a new angle θ' . Then the path of the new ray is

$$x = x_L + z \,\theta'. \tag{45}$$

That is

$$x = h + d\theta + z\theta'. \tag{46}$$



Now we need to figure out what θ' is. At the left hand surface of the lens, the angle of incidence is

$$\theta_i = \theta + \frac{x_L}{R_1}.\tag{47}$$

This follows from the fact that the surface is a sphere with radius R_1 . Call the angle with respect to the z-axis of the ray in the glass θ'' . Then the transmission angle is

$$\theta_t = \theta'' + \frac{x_L}{R_1}.\tag{48}$$

(49)

Snell's Law gives $n\theta'' = \theta$, so



Now we come to the right hand surface. The angle of incidence at this surface is T_{τ}

$$\theta_i = \frac{x_L}{R_2} - \theta''. \tag{50}$$

The transmission angle is

$$\theta_t = \frac{x_L}{R_2} - \theta'. \tag{51}$$

Snell's Law says $n\theta_i = \theta_t$. (Remember that here the transmitted wave is in a medium with index of refraction 1). Thus

$$\theta' = n\theta'' - (n-1)\frac{x_L}{R_2}.$$
(52)



Substituting what we had for $n\theta''$, this is

$$\theta' = \theta - (n-1)\frac{x_L}{R_1} - (n-1)\frac{x_L}{R_2}.$$
(53)

Suppose that just for convenience we define

$$(n-1)\left(\frac{1}{R_1} + \frac{1}{R_1}\right) \equiv \frac{1}{f}.$$
(54)

Then our result can be expressed as

$$\theta' = \theta - \frac{x_L}{f}.\tag{55}$$

We can substitute this into Eq. (46),

$$x = h + d\theta + z \left[\theta - \frac{x_L}{f}\right].$$
(56)

We also need to substitute the definition (44) for x_L into this, giving

$$x = h - z\frac{h}{f} + \left\{d + z\left[1 - \frac{d}{f}\right]\right\}\theta.$$
(57)

Now, what do we make of this? We have the equation for the (new) path followed by the ray that started life at angle θ . At a given value of z, the x values are ordinarily all different. That is, x depends on θ for fixed z. However, for one value of z there is a single x independent of θ . This z is the z where the image is, and the corresponding x is the x where the image is. Let us call the z where the image is d'. Then

$$0 = \left\{ d + d' \left[1 - \frac{d}{f} \right] \right\}$$
(58)

or

$$d + d' = \frac{dd'}{f}.$$
(59)

This is usually written as

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f}.$$
(60)

This is known as the *thin lens equation*. The length f is called the focal length of the lens. Recall that f is given by Eq. (54).

Let us call the x coordinate of the image h'. Then

$$h' = h \left[1 - \frac{d'}{f} \right]$$
$$= h d' \left[\frac{1}{d'} - \frac{1}{f} \right]$$
$$= -h \frac{d'}{d}.$$
 (61)

The minus sign indicates that, for an extended object, the extended image is upside down.

Books on general physics usually have a derivation of the thin lens equation, but usually the derivation doesn't directly cover the case that h is not zero. You should read what your general physics book has to say.

The magnification equation is easy to use. It just says that 1) the image is upside down in the simple case that d and d' are positive and 2) the image is magnified compared to the object by the ratio of d' to d. There is a geometric way to see this. One of the rays you could draw starts from the object and passes through the center of the lens. This ray meets a vertical surface and is bent some, then meets another vertical surface and is bent back in the original direction. Thus this particular ray is not bent. The ray then passes through the image. If you draw this, you will see that there are similar triangles that give the relation h/d = h'/d'.

The thin lens equation (60) is also simple to use. A special case is $d = \infty$, which gives d' = f. Rays directed parallel to the lens axis are brought to a

focus at the focal point of the lens. For $f < d < \infty$ there is a real image with $f < d' < \infty$. As d decreases, d' increases. When d = f, the image is at infinity. For 0 < d < f we have $-\infty < d' < 0$; the image is on the left hand side of the lens and is a virtual image.

It is also possible to make a lens that is concave instead of convex. The formulas still work: just take R_1 and R_2 to be negative. Then f is negative. With f < 0, the image distance d' is always negative (for positive d) and the image is virtual.

Problem 5.1 A double concave lens of index of refraction 1.45 has radii of magnitude 30 cm and 25 cm. An object is located 80 cm to the left of the lens. Find (a) the focal length of the lens, (b) the location of the image, and (c) the magnification of the image. (d) Is the image real or virtual? Upright or inverted?

Sometimes it is helpful to draw a ray diagram using special rays that are extra helpful. The first of these is a ray from the object through the center of the lens. This ray is not deflected. The second is a ray coming from the object parallel to the lens axis. This ray is deflected and passes through the focal point on the right. The third is a ray coming from the object and passing through the focal point on the left. This ray is deflected and emerges directed parallel to the lens axis.

Problem 5.2 An object 3.0 cm high is placed 20 cm in front of a thin lens of power 20 diopters. Draw a precise ray diagram to find the position and size of the image and check your result using the thin-lens equation. [The "power" of a lens is the inverse of the focal length and is usually measured in units of diopters, with 1 diopter = 1 m⁻¹. If you purchase glasses, their strength will be specified by giving their power in diopters.]

6 Images as objects

If you make an optical system using two lenses, the first lens can make an image that serves as the object for the second lens. Then the second lens can form an image. You can work out where the final image is by applying the thin lens equation twice. A good picture helps.



It can happen that the image formed from the first lens is a virtual image. That's fine. Then it is a virtual object for the second lens.

It can happen that the second lens intercepts the rays before they converge to form a real image. That's also fine. The image can still serve as the object for the second lens. I think that I would also call this a virtual object since the rays don't actually meet. In this geometry, the object is on the "wrong side" of the second lens. Then in the thin lens formula the object distance is negative. Again, a good picture helps.

Problem 6.1 Show that a diverging lens can never form a real image from a real object. (*Hint*: show that d' is always negative.) On the other hand, a diverging lens can form a real image from a virtual object. For instance, if f = -30 cm and you would like to make a real image 15 cm from the lens, where should the virtual object be? Illustrate this with a picture.

7 The thin concave mirror

One can work out the optics for a spherical mirror in the same spirit as for a lens. Now the incident ray and the reflected ray are on the same side of the plane of the mirror.

We want to see what kind of image our mirror can make. For this purpose, we consider that the mirror is located in the plane z = 0 in our coordinate system. We consider an object at (x, z) = (h, -d). A light ray coming from this object follows a path

$$x = h + (z+d)\theta \tag{62}$$

where θ is the angle of the ray. At the mirror, z = 0, x is

$$x_L = h + d\,\theta. \tag{63}$$

The mirror reflects the ray to a new angle $\theta'.$ Then the path of the new ray is

$$x = x_L + z \,\theta'. \tag{64}$$

That is

$$x = h + d\theta + z\theta'. \tag{65}$$

Simple geometry gives

$$\theta_i = \frac{x_L}{R} - \theta$$

$$\theta_r = \theta' - \frac{x_L}{R}.$$
(66)

Then the law of reflection, $\theta_i = \theta_r$, gives

$$\theta' = \frac{2x_L}{R} - \theta \tag{67}$$

where R is the radius of curvature of the mirror. Thus the path for the reflected ray is

$$x = h + d\theta + z\left(\frac{2x_L}{R} - \theta\right).$$
(68)

Substituting from Eq. (63), we have

$$x = h + d\theta + z\left(2\frac{h + d\theta}{R} - \theta\right) \tag{69}$$

or

$$x = h + 2h\frac{z}{R} + (d - 2\frac{zd}{R} - z)\theta.$$
 (70)



For most values of z, the x coordinate of the ray depends on θ . But there is one special z for which all the rays have the same x. This happens at z = -d' where

$$0 = d + 2\frac{d'd}{R} + d' \tag{71}$$

This has the form

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f}.$$
(72)

where

$$\frac{1}{f} = \frac{2}{R}.\tag{73}$$

The x coordinate of the image is x = h'

$$h' = h - 2h\frac{d'}{R} = hd'\left(\frac{1}{d'} - \frac{2}{R}\right),$$
(74)

or

$$h' = -h \frac{d'}{d}.$$
(75)

Thus we get the same results for a mirror as we had for a lens except that the "normal" place for the image is on the same side of the mirror as the object and the focal length of the mirror is half its radius of curvature.

Problem 7.1 I would like to use a concave makeup mirror of radius of curvature 1.5 m to inspect my nose. How far from the mirror should my face be for the image to be 80 cm from my face?