PH 351 HW 5

(1) ENERGY IN DRIVEN SHM AS FUNCTION OF DRIVING FREQUENCY.

$$W = \underbrace{\tfrac{1}{2} m \dot{x}^2}_{KE} + \underbrace{\tfrac{1}{2} s x^2}_{PE}$$

Assuming $x(t) = A\cos(\omega t + \phi)$, take average values $\langle\,\rangle$

$$\langle W \rangle = \frac{mA^2}{2} \left\langle \omega^2 \sin^2(\omega t + \phi) + \omega_0^2 \cos^2(\omega t + \phi) \right\rangle$$

$$= \frac{mA^2}{4} (\omega^2 + \omega_0^2) \quad \left[\text{as shown in class } \langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = \tfrac{1}{2}\right]$$

$$\langle W \rangle = \frac{m f_0^2}{4} \frac{\omega^2 + \omega_0^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \qquad \text{ugly!!}$$

we could substitute $\omega \ll \omega_0$ to get $\langle W \rangle \sim \dfrac{m f_0^2}{4\omega_0^2}$

$\omega \gg \omega_0$ " " $\langle W \rangle = \dfrac{m f_0^2}{4\omega^2}$

but this is not very informative.

However looking at

$$W = \frac{mA^2}{2} \left(\omega^2 \sin^2(\omega t + \phi) + \omega_0^2 \cos^2(\omega t + \phi) \right)$$

if $\omega \gg \omega_0$ $\quad W \sim \dfrac{mA^2}{2} \omega^2 \sin^2(\omega t + \phi)$ (KE dominates)

and $\omega \ll \omega_0$ $\quad W \sim \dfrac{mA^2}{2} \omega_0^2 \cos^2(\omega t + \phi)$ (PE dominates).

This makes sense. At low ω, the spring does not stretch and KE dominates. At high ω, SPRING STRETCHES & PE DOMINATES.

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(2) LRC CIRCUIT

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = V_0 \cos\omega t \qquad \begin{cases} f_0 = V_0/L = 3\,C\,s^{-2} \\ \omega = \omega_0 = 100\ s^{-1} \\ \gamma = 1\ s^{-1} \end{cases}$$

STEADY STATE SOLUTION IS

$$Q(t) = A\cos(\omega t + \phi) \text{ where } A = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = \boxed{\frac{f_0}{\gamma\omega_0}}$$

we also find that $\tan\phi = \dfrac{-\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = -1 \Rightarrow \phi = -\pi/2$

$$\boxed{\text{So } Q(t) = \frac{f_0}{\gamma\omega_0}\cos(\omega t - \pi/2) \quad \text{AT STEADY STATE}}$$

WE MUST ADD SOLUTIONS TO:

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = 0 \qquad \text{(TRANSIENT SOLUTIONS TO HOMOGENOUS EQN.)}$$

OR $Q(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_f t + \phi_f)$

here $\left(\frac{\gamma}{2}\right)^2 \ll \omega_0^2$ and $\omega_f = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} \approx 100\ s^{-1}$

$$\tan\phi_f = ?$$

INITIAL CONDITIONS

$$Q(0) = 0 = \underbrace{\frac{f_0}{\gamma\omega_0}\cos\left(-\frac{\pi}{2}\right)}_{\text{STEADY STATE}} + A_0\cos\phi_f$$

since $\dfrac{f_0}{\gamma\omega_0} = \dfrac{3\,C\,s^{-1}}{1\cdot 100\ s^{-2}} = 0.03\,C$

$$Q(0) = 0 = 0.03C \cdot \underbrace{\cos(-\pi/2)}_{0} + A_0\cos\phi_f$$

OR $0 = A_0\cos\phi_f$ Since $A_0 \neq 0$, $\cos\phi_f = 0$, $\phi_f = \pm\pi/2$

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<u>current $\dot{Q}(0) = 5\ C/s$</u>

$$\dot{Q}(t) = \frac{d}{dt}\left[\frac{f_0}{\gamma\omega_0}\cos(\omega t - \pi/2) + A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_f t + \phi_f)\right]$$

$$= -\frac{\omega f_0}{\gamma\omega_0}\sin(\omega t - \pi/2) - \frac{\gamma}{2}A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_f t + \phi_f)$$
$$- \omega_f A_0 e^{-\gamma/2 t}\sin(\omega_f t + \phi_f)$$

$$\dot{Q}(0) = -\frac{\cancel{\omega} f_0}{\gamma \cancel{\omega_0}}\underbrace{\sin(-\pi/2)}_{(-1)} - \frac{\gamma}{2}A_0\cos\phi_f - \omega_f A_0 \sin\phi_f$$

$$5\ C/s = \frac{f_0}{\gamma} - \frac{\cancel{\gamma}}{2}A_0\cos\phi_f - \omega_f A_0\sin\phi_f$$

but $\frac{f_0}{\gamma} = \frac{3\ Cs^{-2}}{1\ s^{-1}} = 3\ Cs^{-1}$ and $\cos\phi_f = 0$

$$5\ Cs^{-1} = 3\ Cs^{-1} - 100\ s^{-1} A_0 \sin(\pm\pi/2)$$

$\boxed{\text{conclude } \phi_f = -\pi/2}$ SO THAT $2\ Cs^{-1} = 100\ s^{-1} \cdot A_0$

$$\boxed{A_0 = 0.02\ C}$$

THE FINAL RESULT IS

$$Q(t) = 0.03\cos(\underset{\omega}{100}t - \pi/2) + 0.02\cos(\underset{\omega_f \sim \omega}{100}t - \pi/2)\underset{\text{DAMPING}}{e^{-t/2}}$$

approximately true $(t \ll 1)$

that $Q(t) = 0.05\cos(100t - \frac{\pi}{2})$ $\left\{ \text{check: } Q(0) = 0,\ \dot{Q}(0) = 5\ Cs^{-1} \right.$

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(3.3) $m\ddot{x} + sx = F_0 \sin\omega t$

write as $m\ddot{x} + sx = F_0 e^{i\omega t}$ and later, take $\mathrm{Im}(x(t))$
(IMAGINARY PART !)

let $x(t) = Ce^{i\omega t}$

$$-m\omega^2 Ce^{i\omega t} + sCe^{i\omega t} = F_0 e^{i\omega t}$$

$$(-m\omega^2 + s)\, C e^{i\omega t} = F_0 e^{i\omega t}$$

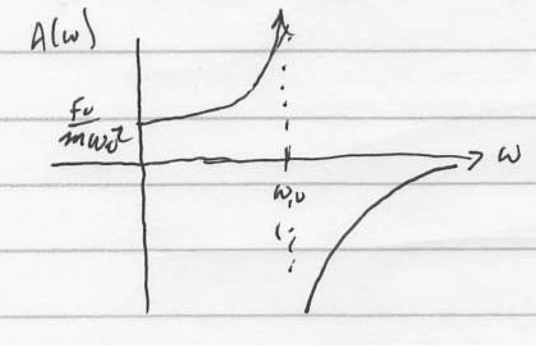
$$C = \frac{F_0}{s - m\omega^2} \quad \text{but} \quad \frac{s}{m} = \omega_0^2$$

so $C = \dfrac{F_0}{m(\omega_0^2 - \omega^2)}$ and $x(t) = \dfrac{F_0}{m(\omega_0^2 - \omega^2)} e^{i\omega t}$

$$\boxed{\text{take solution to be } \mathrm{Im}(x(t)) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \sin\omega t}$$

note that C is real and is the amplitude $A(\omega)$

$$A(\omega) = \frac{F_0}{m(\omega_0^2 - \omega^2)} = \begin{cases} \dfrac{F_0}{m\omega_0^2} & \text{if } \omega \ll \omega_0 \\ \pm\infty & \text{if } \omega \sim \omega_0 \\ -\dfrac{F_0}{m\omega^2} & \text{if } \omega \gg \omega_0 \end{cases}$$



SKETCH

HW5 ⑤:

(3.3) CONTINUED

TO THE ABOVE "PARTICULAR" SOLUTION WE CAN ADD <u>ANY</u> SOLUTION TO THE EQUATION:

$m\ddot{x} + Sx = 0$ FOR WHICH $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$

(A, B TO BE DETERMINED BY INITIAL CONDITIONS)

SO THE COMPLETE SOLUTION IS

$$x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\sin\omega t + A\cos\omega_0 t + B\sin\omega_0 t$$

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(3.4) GIVEN $x(0)=0$ AND $\dot{x}(0)=0$ SOLVE FOR $A + B$

(I) $$x(0) = \frac{F_0}{m(\omega_0^2-\omega^2)}\sin(0) + A\cos(0) + B\sin(0) = 0$$

$$\dot{x}(t) = \frac{\omega F_0}{m(\omega_0^2-\omega^2)}\cos\omega t - \omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)$$

(II) $$\dot{x}(0) = \frac{\omega F_0}{m(\omega_0^2-\omega^2)} + \omega_0 B\cos(0) = 0$$

(I) GIVES $A=0$

(II) GIVES $B = -\frac{\omega}{\omega_0}\frac{F_0}{m(\omega_0^2-\omega^2)}$

SO $$x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\left(\sin\omega_0 t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)$$

(3.4) CONTINUED

now write $\omega = \omega_0 + \Delta\omega$ where $\frac{\Delta\omega}{\omega_0} \ll 1$ and $\Delta\omega t \ll 1$

$$x(t) = \frac{F_0}{m}\,\frac{1}{\omega_0^2 - (\omega_0^2 + \Delta\omega)^2}\left(\sin(\omega_0 + \Delta\omega)t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)$$

(a) denominator becomes $\omega_0^2 - \left(\omega_0^2 + 2\omega_0\Delta\omega + \omega^2\right)$

$$= -2\omega_0\Delta\omega + \Delta\omega^2$$

$$= -2\omega_0\Delta\omega\left(1 + \frac{\Delta\omega}{\omega_0}\right) \simeq -2\omega_0\Delta\omega$$

(b) $\sin(\omega_0 + \Delta\omega)t = \sin\omega_0 t \cos\Delta\omega t + \cos\omega_0 t \sin\Delta\omega t$

but $\begin{cases} \sin\Delta\omega t \sim \Delta\omega t & \text{since } \Delta\omega t \ll 1 \\ \cos\Delta\omega t \sim 1 \end{cases}$

so $\sin(\omega_0 + \Delta\omega)t \cong \sin\omega_0 t + \Delta\omega t\cos\omega_0 t$

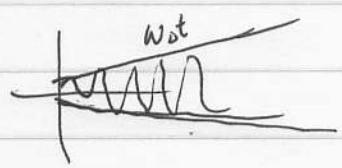
(c) $\frac{\omega}{\omega_0} = \frac{\omega_0 + \Delta\omega}{\omega_0} \cong 1 + \frac{\Delta\omega}{\omega_0}$

$$x(t) = \frac{F_0}{m}\,\frac{1}{-2\omega_0\Delta\omega}\left(\cancel{\sin\omega_0 t} + \Delta\omega t\cos\omega_0 t - \left(\cancel{1} + \frac{\Delta\omega}{\omega_0}\right)\sin\omega_0 t\right)$$

$$= \frac{F_0}{m}\left(\frac{\cancel{\Delta\omega}}{-2\omega_0\cancel{\Delta\omega}}\right)\left(t\cos\omega_0 t - \frac{1}{\omega_0}\sin\omega_0 t\right)$$

(SWITCH SIGNS)

$$= \frac{F_0}{m}\,\frac{1}{2\omega_0^2}\left(\sin\omega_0 t - \omega_0 t\cos\omega_0 t\right) \Rightarrow$$



$$\boxed{\text{GROWS WITHOUT BOUNDS}}$$